

# On ramification in transcendental extensions of local fields

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## Abstract

Let  $L/K$  be an extension of complete discrete valuation fields, and assume that the residue field of  $K$  is perfect and of positive characteristic. The residue field of  $L$  is not assumed to be perfect.

In this paper, we prove a formula for the Swan conductor of the image of a character  $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$  in  $H^1(L, \mathbb{Q}/\mathbb{Z})$  for  $\chi$  sufficiently ramified. Further, we define generalizations  $\psi_{L/K}^{\text{ab}}$  and  $\psi_{L/K}^{\text{AS}}$  of the classical  $\psi$ -function and prove a formula for  $\psi_{L/K}^{\text{ab}}(t)$  for sufficiently large  $t \in \mathbb{R}$ .

## 1 Introduction

Let  $K$  be a complete discrete valuation field. Classical ramification theory has extensively studied finite Galois extensions  $L/K$  when the residue field of  $K$  is perfect. Much progress has also been achieved when the residue field is no longer assumed to be perfect, such as K. Kato's generalization of the classical Swan conductor  $\text{Sw } \chi \in \mathbb{Z}_{\geq 0}$  for abelian characters  $\chi : G(L/K) \rightarrow \mathbb{Q}/\mathbb{Z}$  ([4]) and A. Abbes and T. Saito's generalization of the upper ramification filtration  $G(L/K)$  ([1]). Yet there are still many open questions, both when the residue field of  $K$  is imperfect and when the extension  $L/K$  is transcendental.

Let  $L/K$  be a finite Galois extension of complete discrete valuation fields with perfect residue fields. Denote by  $e(L/K)$  the ramification index of  $L/K$  and by  $D_{L/K}^{\log}$  the wild different of  $L/K$ , i.e.,  $D_{L/K}^{\log} = D_{L/K} - e(L/K) + 1$ , where  $D_{L/K}$  is the different of  $L/K$ . It's classically known that, if  $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$  and  $\chi_L$  is its image in  $H^1(L, \mathbb{Q}/\mathbb{Z})$ , then, when  $\text{Sw } \chi \gg 0$ ,

$$\text{Sw } \chi_L = \psi_{L/K}(\text{Sw } \chi) = e(L/K) \text{Sw } \chi - D_{L/K}^{\log}, \quad (1.1)$$

where  $\psi_{L/K}$  is the classical  $\psi$ -function (see, for example, [9]).

In this paper, we obtain a formula resembling (1.1) for (possibly transcendental) extensions  $L/K$  of complete discrete valuation fields when the residue field of  $K$  is perfect but the residue field of  $L$  is not necessarily perfect, and then define generalizations of the classical

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$\psi$ -function. To be precise, we first prove the two following results, the first when  $L$  is of equal positive characteristic and the second when  $L$  is of mixed characteristic. Here  $\hat{\Omega}_{\mathcal{O}_L/\mathcal{O}_K}^1(\log)$  denotes the completed  $\mathcal{O}_L$ -module of relative differential forms with log poles,  $\delta_{\text{tor}}(L/K)$  the length of its torsion part, and  $e_K$  the absolute ramification index of  $K$ . For a character  $\chi \in H^1(L, \mathbb{Q}/\mathbb{Z})$ ,  $\text{Sw } \chi$  denotes Kato's Swan conductor of  $\chi$  (defined in [4]).

**Main Result 1** (Theorem 2.11). *Let  $L/K$  be an extension of complete discrete valuation fields of equal characteristic  $p > 0$ . Assume that  $K$  has perfect residue field and  $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$  is such that*

$$\text{Sw } \chi > \frac{p}{p-1} \frac{\delta_{\text{tor}}(L/K)}{e(L/K)}.$$

*Denote by  $\chi_L$  its image in  $H^1(L, \mathbb{Q}/\mathbb{Z})$ . Then*

$$\text{Sw } \chi_L = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L/K).$$

**Main Result 2** (Theorem 4.13). *Let  $L/K$  be an extension of complete discrete valuation fields of mixed characteristic. Assume that  $K$  has perfect residue field of characteristic  $p > 0$  and  $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$  is such that*

$$\text{Sw } \chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil.$$

*Denote by  $\chi_L$  its image in  $H^1(L, \mathbb{Q}/\mathbb{Z})$ . Then*

$$\text{Sw } \chi_L = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L/K).$$

After proving these two main results, we relate this discussion to the  $\psi_{L/K}$  function for  $L/K$ . More precisely, we define two  $\psi$ -functions  $\psi_{L/K}^{\text{AS}}$  and  $\psi_{L/K}^{\text{ab}}$  when  $K$  has perfect residue field but  $L$  has residue field not necessarily perfect. We then show that, in the classical case of finite  $L/K$ , both these definitions coincide with the classical  $\psi_{L/K}$  function. Finally, we prove that we can regard our first two main theorems as formulas for  $\psi_{L/K}^{\text{ab}}(t)$  for  $t \gg 0$ :

**Main Result 3** (Theorem 5.4). *Let  $L/K$  be an extension of complete discrete valuation fields. Assume that  $K$  has perfect residue field of characteristic  $p > 0$ . Let  $t \in \mathbb{R}_{\geq 0}$  be such that*

$$\begin{cases} t \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil & \text{if } K \text{ is of characteristic } 0, \\ t > \frac{p}{p-1} \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} & \text{if } K \text{ is of characteristic } p. \end{cases}$$

*Then*

$$\psi_{L/K}^{\text{ab}}(t) = e(L/K)t - \delta_{\text{tor}}(L/K).$$

Our methods for the proof of Main Result 1 differ greatly from those for the proof of Main Result 2. In the equal characteristic case, we use Artin-Schreier-Witt theory. In the mixed characteristic case, we use M. Kurihara's exponential map ([5]) and a modified version of higher dimensional local class field theory.

We hope to apply the results of this paper to generalize a previous work ([6]) that studies the ramification of the action of the absolute Galois group  $G_K$  on  $H^1(U_{\overline{K}}, \mathcal{G})$  (where  $U$  is an open curve over  $K$  and  $\mathcal{G}$  a smooth  $\ell$ -adic sheaf of rank 1 on  $U$ ) from the semi-stable case to a more general one.

The organization of this paper is the following: in Section 2, we study the positive characteristic case and prove Main Result 1. In Section 3, we introduce the discussion of the mixed characteristic case by studying the example of a two-dimensional local field whose last residue field is finite. In Section 4, we study the general mixed characteristic case and prove Main Result 2. In Section 5, we define generalizations of the  $\psi$ -function when the residue field of  $L$  is not necessarily perfect and prove Main Result 3, which connects them with the other main results of this paper.

**Notation.** Through this paper, for a complete discrete valuation field  $K$ ,  $\mathcal{O}_K$  denotes its ring of integers,  $\mathfrak{m}_K$  the maximal ideal,  $\pi_K$  a prime element, and  $G_K$  the absolute Galois group. Lowercase  $k$  denotes the residue field of  $K$ , and  $v_K$  the discrete valuation. We write  $U_K^n = 1 + \mathfrak{m}_K^n$ .

Following the notation in [4], we write

$$H^q(K) = H^q(\mathrm{Spec} K, \mathbb{Q}/\mathbb{Z}(q-1)).$$

When we say that  $K$  is a local field, we mean that  $K$  is a complete discrete valuation field with perfect (not necessarily finite) residue field. Similarly, when we say  $K$  is a  $q$ -dimensional local field, we mean that there is a chain of fields  $K = K_q, K_{q-1}, \dots, K_1, K_0$  such that, for each  $1 \leq i \leq q$ ,  $K_i$  is a complete discrete valuation field with residue field  $K_{i-1}$  and  $K_0$  is a perfect field. When the last residue field  $K_0$  is finite, we say that  $K$  is a  $q$ -dimensional local field with finite last residue field.

We write

$$\hat{\Omega}_{\mathcal{O}_K}^1(\log) = \varprojlim_m \Omega_{\mathcal{O}_K}^1(\log) / \mathfrak{m}_K^m \Omega_{\mathcal{O}_K}^1(\log),$$

where

$$\Omega_{\mathcal{O}_K}^1(\log) = (\Omega_{\mathcal{O}_K}^1 \oplus (\mathcal{O}_K \otimes_{\mathbb{Z}} K^\times)) / (da - a \otimes a, a \in \mathcal{O}_K, a \neq 0).$$

We shall denote by  $P_{\mathrm{tor}}$  the torsion part of an abelian group  $P$ . Let  $L/K$  an extension of complete discrete valuation fields (of either mixed characteristic or positive characteristic  $p > 0$ ). Throughout this paper,  $\delta_{\mathrm{tor}}(L/K)$  shall denote the length of  $\frac{\hat{\Omega}_{\mathcal{O}_L}^1(\log)_{\mathrm{tor}}}{\mathcal{O}_L \otimes_{\mathcal{O}_K} \hat{\Omega}_{\mathcal{O}_K}^1(\log)_{\mathrm{tor}}}$ , and  $e(L/K)$  the ramification index of  $L/K$ . The absolute ramification index of  $K$  shall be denoted by  $e_K$ .

The  $r$ -th Milnor  $K$ -group of  $L$  shall be denoted by  $K_r(L)$ . We denote by  $U^n K_r(L)$  the subgroup of  $K_r(L)$  generated by elements  $\{a, b_1, \dots, b_{r-1}\}$  where  $a \in U_L^n$ ,  $b_i \in L^\times$ , and we write

$$\hat{K}_r(L) = \varprojlim_n K_r(L) / U^n K_r(L)$$

and

$$U^n \hat{K}_r(L) = \varprojlim_{n'} U^n K_r(L) / U^{n'} K_r(L).$$

## 2 Swan conductor in positive characteristic

Let  $L$  be complete discrete valuation field of equal characteristic  $p > 0$ . In this section, we will study extensions  $L/K$  where  $K$  is a local field (and therefore  $k$  is perfect). To be precise, we shall show that, if  $\chi \in H^1(K)$  has Swan conductor sufficiently large, then

$$\text{Sw } \chi_L = e \text{Sw } \chi - \delta_{\text{tor}}(L/K),$$

where  $\chi_L$  is the image of  $\chi$  in  $H^1(L)$  and  $e = e(L/K)$ . For that goal, we will use valuations on differential forms and Witt vectors, as well as the notion of a Witt vector being “best”, defined later.

First of all, we review some concepts necessary for our discussion. By completed free  $\mathcal{O}_L$ -module with basis  $\{e_\lambda\}_{\lambda \in \Lambda}$ , we mean  $\varprojlim_m M/\mathfrak{m}_L^m M$ , where  $M$  is the free  $\mathcal{O}_L$ -module with basis  $\{e_\lambda\}_{\lambda \in \Lambda}$ . Write  $L = l((\pi_L))$  for some prime  $\pi_L \in L$ , where  $l$  is the residue field of  $L$ . Let  $\{b_\lambda\}_{\lambda \in \Lambda}$  be a lift of a  $p$ -basis of  $l$  to  $\mathcal{O}_L$ . Then  $\hat{\Omega}_{\mathcal{O}_L}^1(\log)$  is the completed free  $\mathcal{O}_L$ -module with basis  $\{db_\lambda, d \log \pi_L\}_{\lambda \in \Lambda}$ . Write  $\hat{\Omega}_L^1 = L \otimes_{\mathcal{O}_L} \hat{\Omega}_{\mathcal{O}_L}^1(\log)$ .

Observe that, when  $K$  is a local field of positive characteristic,  $\hat{\Omega}_{\mathcal{O}_K}^1(\log)$  is torsion, and therefore, for an extension of complete discrete valuation fields  $L/K$ ,  $\delta_{\text{tor}}(L/K)$  is simply the length of the torsion part of  $\hat{\Omega}_{\mathcal{O}_L/\mathcal{O}_K}^1(\log)$ .

Denote by  $W_s(L)$  the Witt vectors of length  $s$ . There is a homomorphism  $d : W_s(L) \rightarrow \hat{\Omega}_L^1$  given by

$$a = (a_{s-1}, \dots, a_0) \mapsto \sum_i a_i^{p^i-1} da_i.$$

**Remark 2.1.** In the literature, the operator  $d : W_s(L) \rightarrow \hat{\Omega}_L^1(\log)$  is often denoted by  $F^{s-1}d$ .

We can define valuations on  $\hat{\Omega}_L^1$  and  $W_s(L)$  as follows. If  $\omega \in \hat{\Omega}_L^1$  and  $a \in W_s(L)$ , let

$$v_L^{\log} \omega = \sup \left\{ n : \omega \in \pi_L^n \otimes_{\mathcal{O}_L} \hat{\Omega}_{\mathcal{O}_L}^1(\log) \right\},$$

and

$$v_L(a) = -\max_i \{-p^i v_L(a_i)\} = \min_i \{p^i v_L(a_i)\}.$$

These valuations define increasing filtrations of  $\hat{\Omega}_L^1$  and  $W_s(L)$  by the subgroups

$$F_n \hat{\Omega}_L^1 = \{\omega \in \hat{\Omega}_L^1 : v_L^{\log} \omega \geq -n\}$$

and

$$F_n W_s(L) = \{a \in W_s(L) : v_L(a) \geq -n\},$$

respectively, where  $n \in \mathbb{Z}_{\geq 0}$ . The latter filtration was defined by Brylinski in [2].

By the theory of Artin-Schreier-Witt, there are isomorphisms

$$W_s(L)/(F-1)W_s(L) \simeq H^1(L, \mathbb{Z}/p^s\mathbb{Z}),$$

where  $F$  is the endomorphism of Frobenius. Kato defined in [4] the filtration  $F_n H^1(L, \mathbb{Z}/p^s\mathbb{Z})$  as the image of  $F_n W_s(L)$  under this map. We recall that, for  $\chi \in H^1(L, \mathbb{Z}/p^s\mathbb{Z})$ , the Swan conductor  $\text{Sw } \chi$  is the smallest  $n$  such that  $\chi \in F_n H^1(L, \mathbb{Z}/p^s\mathbb{Z})$ .

We shall now define what it means for a Witt vector  $a \in W_s(L)$  to be “best”, as well as the notion of relevance length.

**Definition 2.2.** *Let  $a \in W_s(L)$ , and  $n$  be the smallest non-negative integer such that  $a \in F_n W_s(L)$ . We say that  $a$  is best if there is no  $a' \in W_s(L)$  mapping to the same element as  $a$  in  $H^1(L, \mathbb{Z}/p^s\mathbb{Z})$  such that  $a' \in F_{n'} W_s(L)$  for some non-negative integer  $n' < n$ .*

*When  $v_L(a) \geq 0$ ,  $a$  is clearly best. When  $v_L(a) < 0$ ,  $a$  is best if and only if there are no  $a', b \in W_s(L)$  satisfying*

$$a = a' + (F-1)b$$

*and  $v_L(a) < v_L(a')$ .*

We shall start by deducing a simple criterion for determining when  $a$  is best. When  $s = 1$  the characterization of “best  $a$ ” is well-known: every  $a \in \mathcal{O}_L$  is best, and  $a \in L \setminus \mathcal{O}_L$  is best if and only if either  $p \nmid v_L(a)$  or  $p \mid v_L(a)$  but the residue class  $\bar{a} \notin k^p$ . In this section we will characterize best  $a$  for arbitrary  $s$ . We shall prove that  $a$  is best if and only if  $a_i$  is best for some relevant position  $i$ , in the sense of the following definition.

**Definition 2.3.** *We shall say that the  $i$ -th position of  $a$  is relevant if  $v_L(a) = p^i v_L(a_i)$ . Let  $j = \max\{i : v_L(a) = p^i v_L(a_i)\}$ . Then  $j+1$  shall be called the relevance length of  $a$ .*

**Lemma 2.4.** *Let  $a \in W_s(L)$  be of negative valuation. We have  $v_L(a) = v_L^{\log}(da)$  if and only if there is some relevant position  $k$  such that  $v_L(a_k) = v_L^{\log}(da_k)$ .*

*Proof.* Let  $I$  denote the subset of  $\{0, \dots, s-1\}$  consisting of  $i$  such that the  $i$ -th position is relevant and  $v(a_i) = v_L^{\log}(da_i)$ . Let  $j+1$  denote the relevance length of  $a$ . We have

$$da = \sum_{i \in I} a_i^{p^i-1} da_i + \sum_{i \notin I} a_i^{p^i-1} da_i.$$

Clearly

$$v_L^{\log} \left( \sum_{i \notin I} a_i^{p^i-1} da_i \right) > v_L(a),$$

so it is enough to prove that

$$v_L^{\log} \left( \sum_{i \in I} a_i^{p^i-1} da_i \right) = v_L(a)$$

if  $I$  is nonempty.

Assume  $I$  nonempty. Since the relevance length of  $a$  is  $j+1$ ,  $p^j \mid v_L(a)$ . We have  $v_L(a) = -np^j$  for some  $n \in \mathbb{N}$ . For each  $i \in I$ ,  $v_L(a_i) = -np^{j-i}$ . Write  $a_i = \pi_L^{-np^{j-i}} u_i$ , where  $u_i \in \mathcal{O}_L$  is a unit.

Then

$$\sum_{i \in I} a_i^{p^i-1} da_i = \pi_L^{-np^j} \sum_{i \in I} u_i^{p^i-1} du_i - n u_j^{p^j} \frac{d\pi_L}{\pi_L}.$$

If  $p \nmid n$ , then

$$v_L^{\log} \left( \sum_{i \in I} a_i^{p^i-1} da_i \right) = v_L(a).$$

On the other hand, if  $p \mid n$ ,

$$\sum_{i \in I} a_i^{p^i-1} da_i = \pi_L^{-np^j} \sum_{i \in I} u_i^{p^i-1} du_i.$$

Let  $\bar{u}_i$  denote the image of  $u_i$  in the residue field  $l$ . Then

$$v_L^{\log} \left( \pi_L^{-np^j} \sum_{i \in I} u_i^{p^i-1} du_i \right) > v_L(a)$$

if and only if

$$\sum_{i \in I} \bar{u}_i^{p^i-1} d\bar{u}_i = 0.$$

If

$$\sum_{i \in I} \bar{u}_i^{p^i-1} d\bar{u}_i = 0,$$

then, by repeatedly applying the Cartier operator, we see that  $\bar{u}_i \in l^p$  for every  $i \in I$ . This implies  $v_L(a_i) < v_L^{\log}(da_i)$  for every  $i \in I$ , a contradiction. Hence we must have

$$v_L^{\log}(da) = v_L(a). \quad \square$$

**Lemma 2.5.** *Let  $a \in W_s(L)$  be of negative valuation. Assume that  $v_L(a) < v_L^{\log}(da)$  and the relevance length of  $a$  is 1. Then  $a$  is not best.*

*Proof.* Since the relevance length is 1,  $v_L^{\log}(a_i^{p^i-1} da_i) \geq p^i v_L(a_i) > v_L(a_0)$ . Therefore we must have  $v_L(a_0) < v_L^{\log}(da_0)$ , which implies that there exist  $a'_0, b_0 \in L$  such that  $a_0 = a'_0 + b_0^p - b_0$  and  $v_L(a_0) < v_L(a'_0)$ . Let  $a' = (0, \dots, 0, a'_0)$  and  $b' = (0, \dots, 0, b_0)$ . We have

$$a = a' + (F - 1)b,$$

and  $v_L(a) = v_L(a_0) < v_L(a')$ , so  $a$  is not best.  $\square$

**Lemma 2.6.** *Let  $a \in W_s(L)$  be an element of negative valuation. Assume that  $v_L(a) < v_L^{\log}(da)$ . Then  $a$  is not best.*

*Proof.* We shall prove by induction on the relevance length. The case in which  $a$  has relevance length 1 has been proven in Lemma 2.5. Assume now that  $a$  has relevance length  $j + 1$ .

From Lemma 2.4,  $v(a_j) < v_L^{\log}(da_j)$ , so there exist  $a'_j, b_j \in L$  such that  $a_j = a'_j + b_j^p - b_j$  and  $v_L(a_j) < v_L(a'_j)$ . Observe that  $v_L(a_j) = pv_L(b_j)$ . Let  $b = (0, \dots, 0, b_j, 0, \dots, 0)$  and  $a' = a - (F - 1)b$ . Then

$$a' = a - Fb + b = (a_{s-1}, \dots, a_{j+1}, a'_j, \tilde{a}_{j-1}, \dots, \tilde{a}_0),$$

where  $p^i v_L(\tilde{a}_i) \geq v_L(a)$  for every  $0 \leq i \leq j - 1$ .

We have two cases. If  $p^i v_L(\tilde{a}_i) > v_L(a)$  for all  $0 \leq i \leq j - 1$ , then  $v_L(a') > v_L(a)$ , so  $a$  is not best.

On the other hand, if  $v_L(\tilde{a}_i) = v_L(a)$  for some  $0 \leq i \leq j - 1$ , then  $a'$  has relevance length at most  $j$  and  $v_L(a') = v_L(a)$ . Further,  $da' = da + db$ . Since  $v_L(a) < v_L^{\log}(da)$  and  $v_L(a) = pv_L(b) \leq pv_L^{\log}(db)$ , we have  $v_L(a) < v_L^{\log}(da')$ . Thus  $v_L(a') < v_L^{\log}(da')$  and  $a'$  is of relevance length at most  $j$ . By induction,  $a'$  is not best, i.e., there are  $a'', c \in W_s(L)$  such that

$$a' = a'' + (F - 1)c,$$

with  $v(a') < v(a'')$ . Then

$$a = a' + (F - 1)b = a'' + (F - 1)(b + c),$$

with  $v_L(a) < v_L(a'')$ . Thus  $a$  is not best. □

**Theorem 2.7.** *Let  $a \in W_s(L)$ . The following conditions are equivalent:*

- (i)  $a$  is best.
- (ii) There exists some relevant position  $i$  such that  $a_i$  is best in the sense of length one.
- (iii)  $v_L(a) = v_L^{\log}(da)$ .

*Proof.* Observe that, when  $a$  has non-negative valuation, (i), (ii) and (iii) are all simultaneously satisfied, so in the following we assume  $v_L(a) < 0$ .

(ii)  $\Leftrightarrow$  (iii) by Lemma 2.4.

Lemma 2.6 proves (i)  $\Rightarrow$  (iii).

To prove (iii)  $\Rightarrow$  (i), assume that  $a$  is not best. Then there are  $a', b \in W_s(L)$  such that  $a = a' + (F - 1)b$  and  $v_L(a) < v_L(a')$ . We have  $pv_L^{\log}(db) \geq pv_L(b) = v_L(a)$ , so both  $v_L^{\log}(db) > v_L(a)$  and  $v_L^{\log}(da') \geq v_L(a') > v_L(a)$ . Since  $da = da' - db$ , we get that  $v_L^{\log}(da) > v_L(a)$ . □

We shall now use the notion of “best  $a$ ” to construct a homomorphism  $F_n H^1(L, \mathbb{Z}/p^s \mathbb{Z}) \rightarrow F_n \hat{\Omega}_L^1 / F_{[n/p]} \hat{\Omega}_L^1$  satisfying some useful properties. Given an element of  $H^1(L, \mathbb{Z}/p^s \mathbb{Z})$ , it's easy to show the existence of a best  $a \in W_s(L)$  in its preimage. We then have the following proposition:

**Proposition 2.8.**

(i) There is a unique homomorphism

$$\text{rsw} : F_n H^1(L, \mathbb{Z}/p^s \mathbb{Z}) \rightarrow F_n \hat{\Omega}_L^1 / F_{[n/p]} \hat{\Omega}_L^1,$$

called refined Swan conductor, such that the composition

$$F_n W_s(L) \longrightarrow F_n H^1(L, \mathbb{Z}/p^s \mathbb{Z}) \longrightarrow F_n \hat{\Omega}_L^1 / F_{[n/p]} \hat{\Omega}_L^1$$

coincides with

$$d : F_n W_s(L) \rightarrow F_n \hat{\Omega}_L^1 / F_{[n/p]} \hat{\Omega}_L^1.$$

(ii) For  $[n/p] \leq m \leq n$ , the induced map

$$\text{rsw} : F_n H^1(L, \mathbb{Z}/p^s \mathbb{Z}) / F_m H^1(L, \mathbb{Z}/p^s \mathbb{Z}) \rightarrow F_n \hat{\Omega}_L^1 / F_m \hat{\Omega}_L^1$$

is injective.

*Proof.* To prove (i), define  $\text{rsw}$  as follows. Given an element  $\chi \in F_n H^1(L, \mathbb{Z}/p^s \mathbb{Z})$ , take  $a \in F_n W_s(L)$  such that  $a$  is best and the image of  $a$  is  $\chi$ . Then put  $\text{rsw } \chi = da$ .

We must show that this map is well-defined. Let  $a' \in F_n W_s(L)$  be another element that is best and maps to  $\chi$ . Then

$$a = a' + (F - 1)b$$

for some  $b \in W_s(L)$ . We get that  $pv_L^{\log}(db) \geq pv_L(b) \geq -n$ , so  $db \in F_{[n/p]} \hat{\Omega}_L^1$ . Since  $da = da' - db$ , we have that  $da$  and  $da'$  define the same class in  $F_n \hat{\Omega}_L^1 / F_{[n/p]} \hat{\Omega}_L^1$ . Uniqueness of the map is clear.

We shall now prove (ii). Let  $\chi \in F_n H^1(L, \mathbb{Z}/p^s \mathbb{Z})$  such that  $\text{rsw } \chi \in F_m \hat{\Omega}_L^1$ . Take  $a \in F_n W_s(L)$  that is best and such that  $da = \text{rsw } \chi$ . Since  $a$  is best, we have

$$v_L^{\log}(\text{rsw } \chi) = v_L^{\log}(a) = v_L(a) \geq -m,$$

so  $a \in F_m W_s(L)$ . It follows that  $\chi \in F_m H^1(L, \mathbb{Z}/p^s \mathbb{Z})$ .  $\square$

**Remark 2.9.** Our refined Swan conductor  $\text{rsw}$  is a refinement of the refined Swan conductor defined by K. Kato in [4, §5].

Let  $L/K$  be an extension of complete discrete valuation fields of positive characteristic  $p > 0$ , and assume that  $K$  has perfect residue field  $k$ . Let  $\chi \in H^1(K)$  and  $\chi_L$  its image in  $H^1(L)$ . We shall now use Proposition 2.8 to compute the Swan conductor of  $\chi_L$ . We will need the following lemma:

**Lemma 2.10.** *Let  $L/K$  be an extension of complete discrete valuation fields of positive characteristic  $p > 0$ . Write  $e = e(L/K)$  and assume that  $k$  is perfect.*

*Let  $\omega \in \hat{\Omega}_K^1$ , and  $\omega_L$  be the image of  $\omega$  in  $\hat{\Omega}_L^1$ . Then*

$$v_L^{\log}(\omega_L) = ev_K^{\log}(\omega) + \delta_{\text{tor}}(L/K).$$

*Proof.* There exists an exact sequence



$$0 \longrightarrow \mathcal{O}_L \otimes \hat{\Omega}_{\mathcal{O}_K}^1(\log) \longrightarrow \hat{\Omega}_{\mathcal{O}_L}^1(\log) \longrightarrow \hat{\Omega}_{\mathcal{O}_L/\mathcal{O}_K}^1(\log) \longrightarrow 0.$$

Further, since the residue field  $k$  of  $K$  is perfect,  $\hat{\Omega}_{\mathcal{O}_K}^1(\log) = \mathcal{O}_K \frac{d\pi_K}{\pi_K}$ . Let  $\{b_\lambda\}_{\lambda \in \Lambda}$  be a lift of a  $p$ -basis of  $l$  to  $\mathcal{O}_L$ , so that  $\hat{\Omega}_{\mathcal{O}_L}^1(\log)$  is the completed free module with basis  $\{db_\lambda, d \log \pi_L\}_{\lambda \in \Lambda}$ . Write  $\frac{d\pi_K}{\pi_K} = \sum \alpha_\lambda db_\lambda + \alpha d \log \pi_L$ , where  $\alpha_\lambda, \alpha \in \mathcal{O}_L$ . Then  $\delta_{\text{tor}}(L/K) = \min_i \{v_L(e_i)\} = v_L^{\log} \left( \frac{d\pi_K}{\pi_K} \right)$ . Writing  $\omega = \lambda \frac{d\pi_K}{\pi_K}$  for some  $\lambda \in K$ , we see that

$$v_L^{\log}(\omega) = v_L(\lambda) + v_L^{\log} \left( \frac{d\pi_K}{\pi_K} \right) = ev_K(\lambda) + \delta_{\text{tor}}(L/K) = ev_K^{\log}(\omega) + \delta_{\text{tor}}(L/K). \quad \square$$

**Theorem 2.11.** *Let  $L/K$  be an extension of complete discrete valuation fields of equal characteristic  $p > 0$ . Assume that  $K$  has perfect residue field.*

*Denote by  $e(L/K)$  the ramification index of  $L/K$ . Assume that  $\chi \in H^1(K)$  is such that*

$$\text{Sw } \chi > \frac{p}{p-1} \frac{\delta_{\text{tor}}(L/K)}{e(L/K)}.$$

*Let  $\chi_L$  be its image in  $H^1(L)$ . Then*

$$\text{Sw } \chi_L = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L/K).$$

*Proof.* Write  $e = e(L/K)$ . It's enough to show that, for a character  $\chi \in H^1(K, \mathbb{Z}/p^s\mathbb{Z})$  corresponding to the Artin-Schreier-Witt equation  $(F-1)X = a$ , we have that, if  $\text{Sw } \chi > p(p-1)^{-1}e^{-1}\delta_{\text{tor}}(L/K)$ , then

$$\text{Sw } \chi_L = e \text{Sw } \chi - \delta_{\text{tor}}(L/K).$$

To simplify notation, write  $n = \text{Sw } \chi$ ,  $\delta_{\text{tor}} = \delta_{\text{tor}}(L/K)$ . The case  $e = 1$  is simple, so we assume  $e > 1$ . Since  $\text{Sw } \chi > p(p-1)^{-1}e^{-1}\delta_{\text{tor}}(L/K)$ , we have that  $\frac{en}{p} < en - \delta_{\text{tor}}$ , so  $\lfloor \frac{en}{p} \rfloor \leq en - \delta_{\text{tor}} - 1$ . From that, Proposition 2.7, and Lemma 2.10, we get that the diagram

$$\begin{array}{ccc} F_n H^1(K, \mathbb{Z}/p^s\mathbb{Z}) / F_{n-1} H^1(K, \mathbb{Z}/p^s\mathbb{Z}) & \longrightarrow & F_n \hat{\Omega}_K^1 / F_{n-1} \hat{\Omega}_K^1 \\ \downarrow & & \downarrow \\ F_{en} H^1(L, \mathbb{Z}/p^s\mathbb{Z}) / F_{en-\delta_{\text{tor}}-1} H^1(L, \mathbb{Z}/p^s\mathbb{Z}) & \longrightarrow & F_{en} \hat{\Omega}_L^1 / F_{en-\delta_{\text{tor}}-1} \hat{\Omega}_L^1 \end{array}$$

commutes, and the horizontal arrows are injective. Thus

$$\text{Sw } \chi_L = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L/K). \quad \square$$

### 3 The example of a two-dimensional local field of mixed characteristic with finite last residue field

In Section 2, we proved Main Result 1. We shall now focus on proving Main Result 2. Let  $L/K$  be an extension of complete discrete valuation fields of mixed characteristic, and assume that

$K$  has perfect residue field. We will show that, if  $\chi \in H^1(K)$  has Swan conductor sufficiently large, then

$$\text{Sw } \chi_L = e \text{Sw } \chi - \delta_{\text{tor}}(L/K),$$

where  $\chi_L$  is the image of  $\chi$  in  $H^1(L)$  and  $e = e(L/K)$  is the ramification index of  $L/K$ .

The proof of this result is based on two key ideas: the commutativity of a diagram of the form

$$\begin{array}{ccc} \mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(L) \\ \downarrow \text{Res}_{L/K} & & \downarrow \text{Res}_{L/K} \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times \end{array}$$

and a modified version of higher dimensional local class field theory. In order to facilitate the comprehension and illustrate the main ideas, in the present section we will consider, in a brief and expository way, the special case in which  $L$  is a two-dimensional local field with finite last residue field. In this special case, the second key idea is simpler, since we can use two-dimensional local class field theory without any modification. In Section 4 we consider the general case in which  $L$  is a complete discrete valuation field of mixed characteristic.

Through this section, we let  $L$  be a two-dimensional local field of mixed characteristic with residue field  $l$  of characteristic  $p > 0$ , and  $K \leq L$  a one-dimensional local field with finite residue field  $k$ .

As a consequence of [7], there is a residue homomorphism

$$\text{Res}_{L/K} : \hat{\Omega}_{\mathcal{O}_L}^1 \rightarrow \mathcal{O}_K$$

which induces

$$\text{Res}_{L/K} : \hat{\Omega}_{\mathcal{O}_L}^1(\log) \otimes_{\mathcal{O}_L} L \rightarrow K.$$

**Example 3.1.** When  $L = K\{\{T\}\}$  (see page 17),

$$\text{Res}_{L/K} \left( \sum_{i=-\infty}^{\infty} a_i T^i \frac{dT}{T} \right) = a_0.$$

From [5], if  $\eta \in \mathcal{O}_L$  is such that  $v_L(\eta) \geq \frac{2e_L}{p-1} + 1$ , there exists an exponential map

$$\exp_\eta : \hat{\Omega}_{\mathcal{O}_L}^1(\log) \rightarrow \hat{K}_2(L).$$

This map is used in the following theorem, which is the first key step in the proof of the main result for the special case of a two-dimensional local field with finite last residue field. Its proof is omitted due to similarity with that of Theorem 4.11.

**Theorem 3.2.** *Let  $L$  be a two-dimensional local field of mixed characteristic and with finite last residue field, and  $K \leq L$  a local field. Write  $e = e(L/K)$ . Let  $\eta \in \mathcal{O}_K$  be such that*

$$n = v_K(\eta) \geq \frac{2e_K}{p-1} + \frac{1}{e}.$$

Then, if  $n' \in \mathbb{N}$  satisfies

$$n' \geq \frac{\delta_{\text{tor}}(L/K)}{e},$$

we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^1(\log) & \xrightarrow{\exp_\eta} & \hat{K}_2(L) \\ \downarrow \text{Res}_{L/K} & & \downarrow \text{Res}_{L/K} \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times \end{array}$$

where the right vertical arrow is the residue homomorphism from  $K$ -theory defined in [3] and the top and bottom horizontal maps are, respectively, the exponential maps  $\exp_{\eta,2}$  and  $\exp_{\eta,1}$  defined in [5].

We observe that  $\mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^1(\log) \rightarrow \mathfrak{m}_K^{n'}$  in the diagram above is surjective (see Proposition 4.10) and the images of  $\exp_\eta : \mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^1(\log) \rightarrow \hat{K}_2(L)$  and  $\exp_\eta : \mathfrak{m}_K^{n'} \rightarrow K^\times$  are, respectively,  $U^{e(n+n') - \delta_{\text{tor}}(L/K)} \hat{K}_2(L)$  and  $U_K^{n+n'}$  (see Lemma 4.2).

Theorem 3.2 is then combined with two-dimensional local class field theory to prove the main result in the particular case of a two-dimensional local field of mixed characteristic with finite last residue field:

**Theorem 3.3.** *Let  $L$  be a two-dimensional local field of mixed characteristic with finite last residue field, and  $K \leq L$  be a local field. Assume that  $\chi \in H^1(K)$  is such that*

$$\text{Sw } \chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil.$$

Denote by  $\chi_L$  its image in  $H^1(L)$ . Then

$$\text{Sw } \chi_L = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L/K).$$

*Proof.* Write  $e = e(L/K)$ . Let  $n' = \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e} \right\rceil$  and  $n = \text{Sw } \chi - n'$ . Pick  $\eta \in \mathcal{O}_K$  with  $v_K(\eta) = n$ . By two-dimensional local class field theory, the diagram

$$\begin{array}{ccc} \hat{K}_2(L) & \longrightarrow & G_L^{\text{ab}} \\ \downarrow \text{Res}_{L/K} & & \downarrow \\ K^\times & \longrightarrow & G_K^{\text{ab}} \end{array}$$

commutes. Together with Theorem 3.2, this gives us a commutative diagram

$$\begin{array}{ccccc} \mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^1(\log) & \xrightarrow{\exp_\eta} & \hat{K}_2(L) & \longrightarrow & G_L^{\text{ab}} \\ \downarrow \text{Res}_{L/K} & & \downarrow \text{Res}_{L/K} & & \downarrow \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times & \longrightarrow & G_K^{\text{ab}} \end{array}$$

We know that  $\text{Sw } \chi = m$  if and only if  $\chi$  kills  $U_K^{m+1}$  but not  $U_K^m$ , so it follows from the commutative diagram above and Lemma 4.2 that

$$\text{Sw } \chi_L = e(n' + n) - \delta_{\text{tor}}(L/K) = e \text{Sw } \chi - \delta_{\text{tor}}(L/K). \quad \square$$

As a guide for Section 4, we will use Theorem 3.3 to get the same result for a complete discrete valuation field of mixed characteristic  $L$  which has residue field that is a function field in one variable over a finite field. In Section 4, Proposition 4.12 will be used to obtain Theorem 4.13 in an analogous way.

**Corollary 3.4.** *Let  $L$  be a complete discrete valuation field of mixed characteristic, and  $K \leq L$  be a local field. Assume that the residue field  $l$  of  $L$  is a function field in one variable over the finite residue field  $k$  of  $K$ .*

*Assume that  $\chi \in H^1(K)$  is such that*

$$\text{Sw } \chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil.$$

*Denote by  $\chi_L$  its image in  $H^1(L)$ . Then*

$$\text{Sw } \chi_L = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L/K).$$

*Proof.* It is sufficient to prove that this case can be reduced to that of a two-dimensional local field with finite last residue field.

Since  $l$  is a function field in one variable over  $k$ ,  $l$  is a finite separable extension of  $k(T)$  for some transcendental element  $T$ . Then there is an embedding of  $l$  into a finite separable extension  $E$  of  $k((T))$ . Note that  $\{T\}$  is a  $p$ -basis for both  $l$  and  $E$ . Then there is a complete discrete valuation field  $L(E)$  which is an extension of  $L$  satisfying  $\mathcal{O}_L \subset \mathcal{O}_{L(E)}$ ,  $\mathcal{O}_{L(E)} \mathfrak{m}_L = \mathfrak{m}_{L(E)}$ , and the residue field of  $L(E)$  is isomorphic to  $E$  over  $l$ .

From [4, Lemma 6.2], we get that  $\text{Sw } \chi_{L(E)} = \text{Sw } \chi_L$ . Further, since  $E$  is a one-dimensional local field,  $L(E)$  is a two-dimensional local field. Finally, since  $\pi_L$  is a prime in  $L(E)$ ,  $\hat{\Omega}_{\mathcal{O}_{L(E)}/\mathcal{O}_L}^1(\log)$  is torsion free. Then we get, from Lemma 4.3,  $\delta_{\text{tor}}(L/K) = \delta_{\text{tor}}(L(E)/K)$ .

Thus it is sufficient to prove that

$$\text{Sw } \chi_{L(E)} = e(L(E)/K) \text{Sw } \chi - \delta_{\text{tor}}(L(E)/K),$$

which follows from Theorem 3.3.  $\square$

## 4 Swan conductor in the general mixed characteristic case

In this section, we shall generalize the results of the previous section to the more general case in which  $L$  is any complete discrete valuation field of mixed characteristic. We start by briefly reviewing some necessary background and proving some preliminary results.

Let  $L$  be a complete discrete valuation field of mixed characteristic. Let  $B$  be a lift of a  $p$ -basis of the residue field  $l$  to  $\mathcal{O}_L$ . Write  $\{e_\lambda\}_{\lambda \in \Lambda} = \{db : b \in B\} \cup \{d \log \pi_L\}$ . The  $\mathcal{O}_L$ -module  $\hat{\Omega}_{\mathcal{O}_L}^1(\log)$  has the structure

$$\hat{M} \oplus \mathcal{O}_L / \mathfrak{m}_L^a \mathcal{O}_L$$

for some  $a \in \mathbb{Z}_{\geq 0}$ . Here  $\hat{M}$  is the completed free  $\mathcal{O}_L$ -module with basis  $\{e_\lambda\}_{\lambda \in \Lambda - \{\mu\}}$ , i.e.,  $\hat{M} = \varprojlim_m M / \mathfrak{m}_L^m M$  where  $M$  is the free  $\mathcal{O}_L$ -module with basis  $\{e_\lambda\}_{\lambda \in \Lambda - \{\mu\}}$  for some  $\mu \in \Lambda$ .

We have, from [5, Theorem 0.1], the existence of an exponential map

$$\exp_{\eta, r+1} : \hat{\Omega}_{\mathcal{O}_L}^r(\log) \rightarrow \hat{K}_{r+1}(L)$$

when  $\eta \in \mathcal{O}_L$  satisfies

$$v_L(\eta) \geq \frac{2e_L}{p-1} + 1.$$

This exponential map satisfies

$$a \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_r}{b_r} \mapsto \{\exp(\eta a), b_1, \dots, b_r\}$$

for  $a \in \mathcal{O}_L$ ,  $b_i \in \mathcal{O}_L^\times$ . We shall denote  $\exp_{\eta, r+1}$  simply by  $\exp_\eta$  through this paper.

**Remark 4.1.** More precisely, in [5], M. Kurihara proved the existence of an exponential map

$$\exp_{\eta, r+1} : \hat{\Omega}_{\mathcal{O}_L}^r \rightarrow \hat{K}_{r+1}(L)$$

when  $\eta \in \mathcal{O}_L$  satisfies

$$v_L(\eta) \geq \frac{2e_L}{p-1}.$$

Considering the existence of a map  $\hat{\Omega}_{\mathcal{O}_L}^r(\log) \rightarrow \hat{\Omega}_{\mathcal{O}_L}^r$  satisfying the commutative diagram

$$\begin{array}{ccc} \hat{\Omega}_{\mathcal{O}_L}^r(\log) & \xrightarrow{\pi_L} & \hat{\Omega}_{\mathcal{O}_L}^r \\ \uparrow & \nearrow \pi_L & \\ \hat{\Omega}_{\mathcal{O}_L}^r & & \end{array}$$

we can define, for

$$v_L(\eta) \geq \frac{2e_L}{p-1} + 1,$$

an exponential map

$$\exp_{\eta, r+1}^{\log} : \hat{\Omega}_{\mathcal{O}_L}^r(\log) \rightarrow \hat{K}_{r+1}(L)$$

by taking the composite

$$\exp_{\eta, r+1}^{\log} = \exp_{\frac{\eta}{\pi_L}, r+1} \circ \pi_L.$$

Through this paper, we omit the superscript log when we write this exponential map.

**Lemma 4.2.** *Let  $L$  be a complete discrete valuation field of mixed characteristic, with residue field  $l$  of characteristic  $p > 0$ . Assume that  $\eta \in \mathcal{O}_L$  satisfies*

$$n = v_L(\eta) \geq \frac{2e_L}{p-1} + 1.$$

*Then the image of the exponential map*

$$\exp_\eta : \mathfrak{m}_L^{n'} \hat{\Omega}_{\mathcal{O}_L}^r(\log) \rightarrow \hat{K}_{r+1}(L)$$

*is  $U^{n+n'} \hat{K}_{r+1}(L)$ .*

*Proof.* Let  $a \in \mathfrak{m}_L^{n'}$ ,  $b_i \in \mathcal{O}_L^\times$ . Observe that

$$a \frac{d\pi_L}{\pi_L} \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{r-1}}{b_{r-1}} \mapsto \{\exp(pa\eta), \pi_L, b_1, \dots, b_{r-1}\}$$

and

$$a \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_r}{b_r} \mapsto \{\exp(a\eta), b_1, \dots, b_r\}.$$

Then the image is contained in  $U^{n+n'} \hat{K}_{r+1}(L)$ . Let  $\tilde{n} \geq n + n'$ . Observe that the maps

$$\frac{\mathfrak{m}_L^{\tilde{n}}}{\mathfrak{m}_L^{\tilde{n}+1}} \otimes \Omega_{\mathcal{O}_L}^r(\log) \rightarrow U^{\tilde{n}} K_{r+1}(L) / U^{\tilde{n}+1} K_{r+1}(L)$$

given by

$$\alpha \otimes \beta \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_r}{b_r} \mapsto \{1 + \alpha\beta, b_1, \dots, b_r\},$$

where  $\alpha \in \mathfrak{m}_L^{\tilde{n}}$ ,  $\beta \in \mathcal{O}_L$ ,  $b_i \in L^\times$ , are surjective. Passing to the limit, we get that  $\exp_\eta : \mathfrak{m}_L^{n'} \hat{\Omega}_{\mathcal{O}_L}^r(\log) \rightarrow U^{n+n'} \hat{K}_{r+1}(L)$  is surjective.  $\square$

We shall now construct some tools and intermediate steps necessary for the obtainment of the main result. Recall that  $\delta_{\text{tor}}(L/K)$  denotes the length of

$$\frac{\hat{\Omega}_{\mathcal{O}_L}^1(\log)_{\text{tor}}}{\mathcal{O}_L \otimes_{\mathcal{O}_K} \hat{\Omega}_{\mathcal{O}_K}^1(\log)_{\text{tor}}}.$$

We have the following property:

**Lemma 4.3.** *Let  $L_2/L_1/L_0$  be a tower of complete discrete valuation fields. Then*

$$\delta_{\text{tor}}(L_2/L_0) = \delta_{\text{tor}}(L_2/L_1) + e(L_2/L_1) \delta_{\text{tor}}(L_1/L_0).$$

*Proof.* Denote by  $n_2$  the length of  $\hat{\Omega}_{\mathcal{O}_{L_2}}^1(\log)_{\text{tor}}$ ,  $n_1$  the length of  $\hat{\Omega}_{\mathcal{O}_{L_1}}^1(\log)_{\text{tor}}$ , and  $n_0$  the length of  $\hat{\Omega}_{\mathcal{O}_{L_0}}^1(\log)_{\text{tor}}$ . We have exact sequences

$$0 \rightarrow \mathcal{O}_{L_2} \otimes_{\mathcal{O}_{L_1}} \hat{\Omega}_{\mathcal{O}_{L_1}}^1(\log)_{\text{tor}} \rightarrow \hat{\Omega}_{\mathcal{O}_{L_2}}^1(\log)_{\text{tor}} \rightarrow \frac{\hat{\Omega}_{\mathcal{O}_{L_2}}^1(\log)_{\text{tor}}}{\mathcal{O}_{L_2} \otimes_{\mathcal{O}_{L_1}} \hat{\Omega}_{\mathcal{O}_{L_1}}^1(\log)_{\text{tor}}} \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}_{L_1} \otimes_{\mathcal{O}_{L_0}} \hat{\Omega}_{\mathcal{O}_{L_0}}^1(\log)_{\text{tor}} \rightarrow \hat{\Omega}_{\mathcal{O}_{L_1}}^1(\log)_{\text{tor}} \rightarrow \frac{\hat{\Omega}_{\mathcal{O}_{L_1}}^1(\log)_{\text{tor}}}{\mathcal{O}_{L_1} \otimes_{\mathcal{O}_{L_0}} \hat{\Omega}_{\mathcal{O}_{L_0}}^1(\log)_{\text{tor}}} \rightarrow 0,$$

and

$$0 \rightarrow \mathcal{O}_{L_2} \otimes_{\mathcal{O}_{L_0}} \hat{\Omega}_{\mathcal{O}_{L_0}}^1(\log)_{\text{tor}} \rightarrow \hat{\Omega}_{\mathcal{O}_{L_2}}^1(\log)_{\text{tor}} \rightarrow \frac{\hat{\Omega}_{\mathcal{O}_{L_2}}^1(\log)_{\text{tor}}}{\mathcal{O}_{L_2} \otimes_{\mathcal{O}_{L_0}} \hat{\Omega}_{\mathcal{O}_{L_0}}^1(\log)_{\text{tor}}} \rightarrow 0$$

which give

$$n_2 = e(L_2/L_1)n_1 + \delta_{\text{tor}}(L_2/L_1),$$

$$n_1 = e(L_1/L_0)n_0 + \delta_{\text{tor}}(L_1/L_0),$$

and

$$n_2 = e(L_2/L_0)n_0 + \delta_{\text{tor}}(L_2/L_0).$$

Combining the first two equations we get

$$n_2 = e(L_2/L_0)n_0 + e(L_2/L_1)\delta_{\text{tor}}(L_1/L_0) + \delta_{\text{tor}}(L_2/L_1).$$

Therefore

$$n_2 = e(L_2/L_0)n_0 + \delta_{\text{tor}}(L_2/L_0) = e(L_2/L_0)n_0 + e(L_2/L_1)\delta_{\text{tor}}(L_1/L_0) + \delta_{\text{tor}}(L_2/L_1),$$

which implies

$$\delta_{\text{tor}}(L_2/L_0) = \delta_{\text{tor}}(L_2/L_1) + e(L_2/L_1)\delta_{\text{tor}}(L_1/L_0).$$

□

**Remark 4.4.** Through this section,  $K$  usually denotes a local field of mixed characteristic, and therefore it has perfect residue field  $k$ . When this is the case, the  $\mathcal{O}_K$ -module  $\hat{\Omega}_{\mathcal{O}_K}^1(\log)$  is a torsion module, and therefore  $\delta_{\text{tor}}(L/K)$  is simply the length of

$$\left( \hat{\Omega}_{\mathcal{O}_L/\mathcal{O}_K}^1(\log) \right)_{\text{tor}}.$$

**Lemma 4.5.** *Let  $L/M$  be a finite extension complete discrete valuation fields of characteristic zero. Assume that the residue field  $l$  of  $L$  has characteristic  $p > 0$  and  $[l : l^p] = p^r$ . Write  $e = e(L/M)$ . Then*

$$\text{Tr}_{L/M} \left( \mathfrak{m}_L^{en - \delta_{\text{tor}}(L/M)} \frac{\hat{\Omega}_{\mathcal{O}_L}^r(\log)}{\hat{\Omega}_{\mathcal{O}_L}^r(\log)_{\text{tor}}} \right) = \mathfrak{m}_M^n \frac{\hat{\Omega}_{\mathcal{O}_M}^r(\log)}{\hat{\Omega}_{\mathcal{O}_M}^r(\log)_{\text{tor}}}$$

and

$$\text{Tr}_{L/M} \left( \mathfrak{m}_L^{en - \delta_{\text{tor}}(L/M) + 1} \frac{\hat{\Omega}_{\mathcal{O}_L}^r(\log)}{\hat{\Omega}_{\mathcal{O}_L}^r(\log)_{\text{tor}}} \right) = \mathfrak{m}_M^{n+1} \frac{\hat{\Omega}_{\mathcal{O}_M}^r(\log)}{\hat{\Omega}_{\mathcal{O}_M}^r(\log)_{\text{tor}}}$$

for every integer  $n$ .

*Proof.* We shall prove the first equality. Let  $\delta(L/M)$  be the length of the  $\mathcal{O}_L$ -module  $\hat{\Omega}_{\mathcal{O}_L/\mathcal{O}_M}^1(\log)$ .

Observe that  $\frac{\hat{\Omega}_{\mathcal{O}_L}^1(\log)}{\hat{\Omega}_{\mathcal{O}_L}^1(\log)_{\text{tor}}}$  and  $\frac{\hat{\Omega}_{\mathcal{O}_M}^1(\log)}{\hat{\Omega}_{\mathcal{O}_M}^1(\log)_{\text{tor}}}$  are free of rank  $r$ . We have an exact sequence

$$0 \rightarrow \mathcal{O}_L \otimes_{\mathcal{O}_M} \frac{\hat{\Omega}_{\mathcal{O}_M}^1(\log)}{\hat{\Omega}_{\mathcal{O}_M}^1(\log)_{\text{tor}}} \rightarrow \frac{\hat{\Omega}_{\mathcal{O}_L}^1(\log)}{\hat{\Omega}_{\mathcal{O}_L}^1(\log)_{\text{tor}}} \rightarrow \frac{\hat{\Omega}_{\mathcal{O}_L}^1(\log)}{\hat{\Omega}_{\mathcal{O}_L}^1(\log)_{\text{tor}}} \Big/ \left( \mathcal{O}_L \otimes_{\mathcal{O}_M} \frac{\hat{\Omega}_{\mathcal{O}_M}^1(\log)}{\hat{\Omega}_{\mathcal{O}_M}^1(\log)_{\text{tor}}} \right) \rightarrow 0.$$

Since the length of

$$\frac{\hat{\Omega}_{\mathcal{O}_L}^1(\log)}{\hat{\Omega}_{\mathcal{O}_L}^1(\log)_{\text{tor}}} \Big/ \left( \mathcal{O}_L \otimes_{\mathcal{O}_M} \frac{\hat{\Omega}_{\mathcal{O}_M}^1(\log)}{\hat{\Omega}_{\mathcal{O}_M}^1(\log)_{\text{tor}}} \right)$$

is  $\delta(L/M) - \delta_{\text{tor}}(L/M)$ , we have that the length of

$$\frac{\hat{\Omega}_{\mathcal{O}_L}^r(\log)}{\hat{\Omega}_{\mathcal{O}_L}^r(\log)_{\text{tor}}} \Big/ \left( \mathcal{O}_L \otimes_{\mathcal{O}_M} \frac{\hat{\Omega}_{\mathcal{O}_M}^r(\log)}{\hat{\Omega}_{\mathcal{O}_M}^r(\log)_{\text{tor}}} \right)$$

is also  $\delta(L/M) - \delta_{\text{tor}}(L/M)$ . Since  $\frac{\hat{\Omega}_{\mathcal{O}_L}^r(\log)}{\hat{\Omega}_{\mathcal{O}_L}^r(\log)_{\text{tor}}}$  and  $\frac{\hat{\Omega}_{\mathcal{O}_M}^r(\log)}{\hat{\Omega}_{\mathcal{O}_M}^r(\log)_{\text{tor}}}$  are both free of rank one,

we have

$$\frac{\hat{\Omega}_{\mathcal{O}_L}^r(\log)}{\hat{\Omega}_{\mathcal{O}_L}^r(\log)_{\text{tor}}} = \mathfrak{m}_L^{\delta_{\text{tor}}(L/M) - \delta(L/M)} \frac{\hat{\Omega}_{\mathcal{O}_M}^r(\log)}{\hat{\Omega}_{\mathcal{O}_M}^r(\log)_{\text{tor}}}.$$

Therefore

$$\begin{aligned} \text{Tr}_{L/M} \left( \mathfrak{m}_L^{en - \delta_{\text{tor}}(L/M)} \frac{\hat{\Omega}_{\mathcal{O}_L}^r(\log)}{\hat{\Omega}_{\mathcal{O}_L}^r(\log)_{\text{tor}}} \right) &= \\ \text{Tr}_{L/M} \left( \mathfrak{m}_L^{en - \delta_{\text{tor}}(L/M)} \mathfrak{m}_L^{\delta_{\text{tor}}(L/M) - \delta(L/M)} \frac{\hat{\Omega}_{\mathcal{O}_M}^r(\log)}{\hat{\Omega}_{\mathcal{O}_M}^r(\log)_{\text{tor}}} \right) &= \\ \text{Tr}_{L/M} \left( \mathfrak{m}_L^{en - \delta(L/M)} \right) \frac{\hat{\Omega}_{\mathcal{O}_M}^r(\log)}{\hat{\Omega}_{\mathcal{O}_M}^r(\log)_{\text{tor}}}. \end{aligned}$$

Let  $\tilde{\delta}(L/M)$  be the length of the  $\mathcal{O}_L$ -module  $\hat{\Omega}_{\mathcal{O}_L/\mathcal{O}_M}^1$ . Since

$$\text{Tr}_{L/M} \left( \mathfrak{m}_L^{e(n+1) - \tilde{\delta}(L/M) - 1} \right) = \mathfrak{m}_M^n$$

and  $\tilde{\delta}(L/M) = \delta(L/M) + e - 1$ , we get

$$\text{Tr}_{L/M} \left( \mathfrak{m}_L^{en - \delta(L/M)} \right) = \mathfrak{m}_M^n.$$

Hence

$$\text{Tr}_{L/M} \left( \mathfrak{m}_L^{en - \delta_{\text{tor}}(L/M)} \frac{\hat{\Omega}_{\mathcal{O}_L}^r(\log)}{\hat{\Omega}_{\mathcal{O}_L}^r(\log)_{\text{tor}}} \right) = \mathfrak{m}_M^n \frac{\hat{\Omega}_{\mathcal{O}_M}^r(\log)}{\hat{\Omega}_{\mathcal{O}_M}^r(\log)_{\text{tor}}}.$$

The second equality is obtained similarly. □



We shall now review  $q$ -dimensional local field (for more on this subject, see [10, 7]). Subsequently, we shall use  $q$ -dimensional local fields to construct some residue maps.

Let  $K$  be a complete discrete valuation field. The field  $K\{\{T\}\}$  is defined as the set

$$K\{\{T\}\} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^i : a_i \in K, \inf v_K(a_i) > -\infty, \text{ and } v_K(a_i) \rightarrow \infty \text{ as } i \rightarrow -\infty \right\}$$

with addition and multiplication as follows:

$$\sum_{i=-\infty}^{\infty} a_i T^i + \sum_{i=-\infty}^{\infty} b_i T^i = \sum_{i=-\infty}^{\infty} (a_i + b_i) T^i$$

and

$$\sum_{i=-\infty}^{\infty} a_i T^i \sum_{i=-\infty}^{\infty} b_i T^i = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_j b_{i-j} T^i.$$

We can define a discrete valuation on  $K\{\{T\}\}$  by setting

$$v_{K\{\{T\}\}} \left( \sum_{i=-\infty}^{\infty} a_i T^i \right) = \min v_K(a_i).$$

Endowed with this valuation,  $K\{\{T\}\}$  becomes a complete discrete valuation field.

When  $K$  is a local field, the field

$$K\{\{T_1\}\} \cdots \{\{T_m\}\}((T_{m+1})) \cdots ((T_{q-1})),$$

where  $1 \leq m \leq q-1$ , is a  $q$ -dimensional local field. Fields of this form are called standard  $q$ -dimensional local fields.

We shall now make the constructions necessary for defining a residue map

$$\text{Res}_{L/K} : \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \rightarrow \mathcal{O}_K$$

for a finite extension  $L$  of the standard  $q$ -dimensional local field  $K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ , where  $K$  is a local field of mixed characteristic.

**Definition 4.6.** *Let  $K$  be a complete discrete valuation field and  $L_0 = K, L_1 = K\{\{T_1\}\}, \dots, L = L_{q-1} = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ . Define*

$$c_{L_i/L_{i-1}} : L_i \rightarrow L_{i-1}$$

by

$$c_{L_i/L_{i-1}} \left( \sum_{k \in \mathbb{Z}} a_k T_i^k \right) = a_0.$$

Then define  $c_{L/K} = c_{L_1/L_0} \circ \cdots \circ c_{L_{q-1}/L_{q-2}}$ .

**Definition 4.7.** Let  $L = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ , where  $K$  is a local field of mixed characteristic. Define the residue map

$$\text{Res}_{L/K} : \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \rightarrow \mathcal{O}_K$$

as the composition

$$\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \rightarrow \hat{\Omega}_{\mathcal{O}_L/\mathcal{O}_K}^{q-1}(\log) \rightarrow \mathcal{O}_K,$$

where  $\hat{\Omega}_{\mathcal{O}_L/\mathcal{O}_K}^{q-1}(\log) \rightarrow \mathcal{O}_K$  is the homomorphism that satisfies, for  $a \in \mathcal{O}_L$ ,

$$ad \log T_1 \wedge \cdots \wedge d \log T_{q-1} \mapsto c_{L/K}(a).$$

It induces

$$\text{Res}_{L/K} : \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \otimes_{\mathcal{O}_L} L \rightarrow K.$$

**Definition 4.8.** Let  $L$  be a finite extension of  $M = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ , where  $K$  is a local field of mixed characteristic. Define the residue map

$$\text{Res}_{L/K} : \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \otimes_{\mathcal{O}_L} L \rightarrow K$$

by

$$\text{Res}_{L/K} = \text{Res}_{M/K} \circ \text{Tr}_{L/M}.$$

We will now start to obtain some properties of the trace and residue maps that will be necessary for the proof of the main theorem of this section.

**Proposition 4.9.** Let  $L$  be a complete discrete valuation field that is a finite extension of  $M = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ , where  $K$  is a local field of mixed characteristic. Write  $e = e(L/K)$ . Then, for any integer  $n$ ,

$$\text{Res}_{L/K} \left( \mathfrak{m}_L^{ne - \delta_{\text{tor}}(L/K)} \frac{\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log)}{\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log)_{\text{tor}}} \right) = \mathfrak{m}_K^n$$

and

$$\text{Res}_{L/K} \left( \mathfrak{m}_L^{ne - \delta_{\text{tor}}(L/K) + 1} \frac{\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log)}{\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log)_{\text{tor}}} \right) = \mathfrak{m}_K^{n+1}.$$

*Proof.* We shall prove the first equality; the second is obtained in a similar way.

Observe that  $\text{Res}_{L/K} = \text{Res}_{M/K} \circ \text{Tr}_{L/M}$ . Since  $\delta_{\text{tor}}(M/K) = 0$ , we have, by Lemma 4.3, that

$$\delta_{\text{tor}}(L/K) = \delta_{\text{tor}}(L/M).$$

Then, using Lemma 4.5, we get

$$\begin{aligned} \text{Res}_{L/K} \left( \mathfrak{m}_L^{ne - \delta_{\text{tor}}(L/K)} \frac{\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log)}{\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log)_{\text{tor}}} \right) &= \text{Res}_{M/K} \left( \text{Tr}_{L/M} \left( \mathfrak{m}_L^{ne - \delta_{\text{tor}}(L/K)} \frac{\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log)}{\hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log)_{\text{tor}}} \right) \right) = \\ &= \text{Res}_{M/K} \left( \mathfrak{m}_M^n \frac{\hat{\Omega}_{\mathcal{O}_M}^{q-1}(\log)}{\hat{\Omega}_{\mathcal{O}_M}^{q-1}(\log)_{\text{tor}}} \right) = \mathfrak{m}_K^n. \end{aligned} \quad \square$$

**Proposition 4.10.** *Let  $L$ ,  $K$ , and  $e$  be as in Proposition 4.9. Then, if  $n \in \mathbb{N}$  satisfies*

$$n \geq \frac{\delta_{\text{tor}}(L/K)}{e},$$

*we have*

$$\text{Res}_{L/K} \left( \mathfrak{m}_L^{ne - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \right) = \mathfrak{m}_K^n$$

*and*

$$\text{Res}_{L/K} \left( \mathfrak{m}_L^{ne - \delta_{\text{tor}}(L/K) + 1} \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \right) = \mathfrak{m}_K^{n+1}.$$

*Proof.* In this case  $en - \delta_{\text{tor}}(L/K) \geq 0$ , so this follows from Proposition 4.9.  $\square$

We will now use the previous properties of residue and trace maps, the exponential map defined by M. Kurihara ([5]), and a modification of higher dimensional class field theory to prove that, when  $L$  is a  $q$ -dimensional local field that is a finite extension of  $K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ , Main Result 2 holds. This will then be used to prove the general result. We start with the following theorem:

**Theorem 4.11.** *Let  $L$  be a  $q$ -dimensional local field that is a finite extension of  $M = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ , where  $K$  is a local field of mixed characteristic with residue field  $k$  of characteristic  $p > 0$ . Denote by  $n_K$  the length of  $\hat{\Omega}_{\mathcal{O}_K}^1(\log)$ , and write  $e = e(L/K)$ . Assume that  $n \in \mathbb{N}$  satisfies*

$$n \geq \frac{2e_K}{p-1} + \frac{1}{e}$$

*and let  $n' \in \mathbb{N}$  be such that  $n' \geq \frac{\delta_{\text{tor}}(L/K)}{e}$ . Take  $\eta \in \mathcal{O}_K$  such that  $v_K(\eta) = n$ .*

*Then we have a commutative diagram*

$$\begin{array}{ccc} \mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(L) \\ \downarrow \text{Res}_{L/K} & & \downarrow \text{Res}_{L/K} \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times \end{array}$$

*where the right vertical arrow is the residue homomorphism from  $K$ -theory defined in [3] and the top and bottom horizontal maps are, respectively, the exponential maps  $\exp_{\eta,q}$  and  $\exp_{\eta,1}$  defined in [5].*

*Proof.* First, observe that the condition

$$n \geq \frac{2e_K}{p-1} + \frac{1}{e}$$

implies

$$en \geq \frac{2e_K e}{p-1} + 1 = \frac{2e_L}{p-1} + 1.$$

Therefore this condition guarantees the convergence of both the top and the bottom exponential maps (by Theorem 0.1 in [5]). Furthermore, the condition

$$n' \geq \frac{\delta_{\text{tor}}(L/K)}{e}$$

guarantees that we can apply Proposition 4.10.

We need to prove that the diagram

$$\begin{array}{ccc}
\mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(L) \\
\downarrow \text{Tr}_{L/M} & & \downarrow N_{L/M} \\
\mathfrak{m}_M^{n'} \hat{\Omega}_{\mathcal{O}_M}^{q-1}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(M) \\
\downarrow \text{Res}_{M/K} & & \downarrow \text{Res}_{M/K} \\
\mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times
\end{array}$$

commutes.

By Proposition 4.10, the map  $\text{Res}_{L/K} : \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \rightarrow \mathcal{O}_K$  induces a surjection

$$\text{Res}_{L/K} : \mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \twoheadrightarrow \mathfrak{m}_K^{n'},$$

and the map  $\text{Res}_{M/K} : \hat{\Omega}_{\mathcal{O}_M}^{q-1}(\log) \rightarrow \mathcal{O}_K$  induces a surjection

$$\text{Res}_{M/K} : \mathfrak{m}_M^{n'} \hat{\Omega}_{\mathcal{O}_M}^{q-1}(\log) \twoheadrightarrow \mathfrak{m}_K^{n'}.$$

A similar argument shows that  $\text{Tr}_{L/M} : \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \rightarrow \hat{\Omega}_{\mathcal{O}_M}^{q-1}(\log)$  induces a surjection

$$\text{Tr}_{L/M} : \mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) \twoheadrightarrow \mathfrak{m}_M^{n'} \hat{\Omega}_{\mathcal{O}_M}^{q-1}(\log).$$

The commutativity of the top square is shown in [5]. The commutativity of the bottom square can be checked explicitly as follows. Let  $M_0 = K, M_1 = K\{\{T_1\}\}, \dots, M_{q-1} = M = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ . It's enough to show that each one of the squares in the diagram

$$\begin{array}{ccc}
\hat{\Omega}_{\mathcal{O}_M}^{q-1}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(M) \\
\downarrow \text{Res}_{M/M_{q-2}} & & \downarrow \text{Res}_{M/M_{q-2}} \\
\hat{\Omega}_{\mathcal{O}_{M_{q-2}}}^{q-2}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_{q-1}(M_{q-2}) \\
\downarrow \text{Res}_{M_{q-2}/M_{q-3}} & & \downarrow \text{Res}_{M_{q-2}/M_{q-3}} \\
\vdots & & \vdots \\
\downarrow \text{Res}_{M_2/M_1} & & \downarrow \text{Res}_{M_2/M_1} \\
\hat{\Omega}_{\mathcal{O}_{M_1}}^1(\log) & \xrightarrow{\exp_\eta} & \hat{K}_2(M_1) \\
\downarrow \text{Res}_{M_1/K} & & \downarrow \text{Res}_{M_1/K} \\
\mathcal{O}_K & \xrightarrow{\exp_\eta} & K^\times
\end{array}$$

commutes. Here the map  $\text{Res}_{M_i/M_{i-1}}$  is the composition

$$\hat{\Omega}_{\mathcal{O}_{M_i}}^i(\log) \rightarrow \hat{\Omega}_{\mathcal{O}_{M_i}/\mathcal{O}_{M_{i-1}}}^i(\log) \rightarrow \hat{\Omega}_{\mathcal{O}_{M_{i-1}}}^{i-1}(\log),$$

where  $\hat{\Omega}_{\mathcal{O}_{M_i}/\mathcal{O}_{M_{i-1}}}^i(\log) \rightarrow \hat{\Omega}_{\mathcal{O}_{M_{i-1}}}^{i-1}(\log)$  is the homomorphism that satisfies

$$ad \log T_1 \wedge \cdots \wedge d \log T_i \mapsto c_{M_i/M_{i-1}}(a) d \log T_1 \wedge \cdots \wedge d \log T_{i-1}$$

for  $a \in \mathcal{O}_{M_i}$ .

Let  $a \in \mathcal{O}_{M_i}$  and write

$$a = \sum_{k < 0} a_k T_i^k + a_0 + \sum_{k > 0} a_k T_i^k,$$

where  $a_k \in \mathcal{O}_{M_{i-1}}$  for every  $k \in \mathbb{Z}$ . Put  $a_- = \sum_{k < 0} a_k T_i^k$  and  $a_+ = \sum_{k > 0} a_k T_i^k$ . Observe first that

$$\begin{aligned} \text{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left( a_0 \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) &= \text{Res}_{M_i/M_{i-1}} \{ \exp(\eta a_0), T_1, \dots, T_i \} = \\ &= \{ \exp(\eta a_0), T_1, \dots, T_{i-1} \} = \exp_\eta \circ \text{Res}_{M_i/M_{i-1}} \left( a_0 \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right). \end{aligned}$$

From Theorem 1 in [3], we have that

$$\text{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left( a_+ \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) = 0 = \exp_\eta \circ \text{Res}_{M_i/M_{i-1}} \left( a_+ \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right).$$

We will now show that we also have

$$\text{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left( a_- \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) = 0 = \exp_\eta \circ \text{Res}_{M_i/M_{i-1}} \left( a_- \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right).$$

From Theorem 1 in [3], we have, for  $k \in \mathbb{Z}_{<0}$  and  $m \in \mathbb{N}$ ,

$$\text{Res}_{M_i/M_{i-1}} \left\{ 1 + \eta a_k T_i^k + \cdots + \frac{(\eta a_k)^m T_i^{mk}}{m!}, T_1, \dots, T_i \right\} = 0.$$

Since  $v_{M_{i-1}}((\eta a_k)^m/m!) \rightarrow \infty$  and the residue map is continuous, we have

$$\text{Res}_{M_i/M_{i-1}} \{ \exp(\eta a_k T_i^k), T_1, \dots, T_i \} = 0.$$

Given  $k \in \mathbb{Z}_{<0}$ , write  $s_k = \sum_{k' \leq k} a_{k'} T_i^{k'}$  and  $a_{-,k} = a_- - s_k$ . The argument above shows that

$$\text{Res}_{M_i/M_{i-1}} \{ \exp(\eta s_k), T_1, \dots, T_i \} = 0.$$

On the other hand, by taking  $-k$  sufficiently large, we can guarantee that

$$\{ \exp(\eta a_{-,k}), T_1, \dots, T_i \} \in U^m \hat{K}_{i+1}(M_i)$$

for arbitrarily large  $m \in \mathbb{N}$ . By continuity of the residue map, we get

$$\text{Res}_{M_i/M_{i-1}} \{ \exp(\eta a_-), T_1, \dots, T_i \} = 0.$$

Hence we conclude that

$$\text{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left( a \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right) = \exp_\eta \circ \text{Res}_{M_i/M_{i-1}} \left( a \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_i}{T_i} \right).$$

A similar argument shows that

$$\begin{aligned} & \text{Res}_{M_i/M_{i-1}} \circ \exp_\eta \left( a \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_{i-1}}{T_{i-1}} \wedge \frac{d\pi_K}{\pi_K} \right) = \\ & \exp_\eta \circ \text{Res}_{M_i/M_{i-1}} \left( a \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_{i-1}}{T_{i-1}} \wedge \frac{d\pi_K}{\pi_K} \right) = 0, \end{aligned}$$

so we conclude that each square in the diagram is commutative.  $\square$

We have now developed all the necessary tools in order to prove Proposition 4.12, which states that Main Result 2 holds when  $L$  is a  $q$ -dimensional local field that is a finite extension of  $K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ . We will then use Proposition 4.12 to prove Theorem 4.13, which gives Main Result 2 in full generality.

**Proposition 4.12.** *Let  $L$  be a  $q$ -dimensional local field that is a finite extension of  $M = K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ , where  $K$  is a local field of mixed characteristic with residue field  $k$  of characteristic  $p > 0$ . Assume that  $\chi \in H^1(K)$  is such that*

$$\text{Sw } \chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil.$$

Denote by  $\chi_L$  its image in  $H^1(L)$ . Then

$$\text{Sw } \chi_L = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L/K).$$

*Proof.* Using the same argument as in [4, (7.6)], we can assume that  $H^1(k) \neq 0$ . Let  $L = L_q, l = L_{q-1}, \dots, L_1, L_0$  be the chain of residue fields of the  $q$ -dimensional local field  $L$ . Since there are isomorphisms

$$H^{q+1}(L)\{p\} \simeq H^q(L_{q-1})\{p\} \simeq H^{q-1}(L_{q-2})\{p\} \simeq \cdots \simeq H^1(L_0)\{p\}$$

and

$$H^2(K)\{p\} \simeq H^1(k)\{p\},$$

we have a commutative diagram

$$\begin{array}{ccccccc} H^1(L) & \times & \hat{K}_q(L) & \xrightarrow{\{\cdot, \cdot\}_L} & H^{q+1}(L)\{p\} & \xrightarrow{\simeq} & H^1(L_0)\{p\} \\ \uparrow & & \downarrow \text{Res}_{L/K} & & & & \downarrow \\ H^1(K) & \times & K^\times & \xrightarrow{\{\cdot, \cdot\}_K} & H^2(K)\{p\} & \xrightarrow{\simeq} & H^1(k)\{p\} \end{array}$$

Here, the pairing  $H^1(L) \times \hat{K}_q(L) \rightarrow H^{q+1}(L)\{p\}$  is the one constructed in [4]. To simplify notation, put  $e = e(L/K)$ ,  $n' = \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e} \right\rceil$  and  $n = \text{Sw } \chi - n'$ . From [4, Proposition 6.5], we have that

$$\text{Sw } \chi_L = m \geq 1$$

if and only if

$$\{\chi_L, U^{m+1} \hat{K}_q(L)\}_L = 0$$

but

$$\{\chi_L, U^m \hat{K}_q(L)\}_L \neq 0.$$

Denote the composition  $H^1(L) \times \hat{K}_q(L) \rightarrow H^{q+1}(L)\{p\} \rightarrow H^1(k)\{p\}$  by  $\{ , \}_k$ . Pick  $\eta \in \mathcal{O}_K$  such that  $v_K(\eta) = n$ . From Lemma 4.2 and the commutative diagram

$$\begin{array}{ccc} \mathfrak{m}_L^{en' - \delta_{\text{tor}}(L/K)} \hat{\Omega}_{\mathcal{O}_L}^{q-1}(\log) & \xrightarrow{\exp_\eta} & \hat{K}_q(L) \\ \downarrow \text{Res}_{L/K} & & \downarrow \text{Res}_{L/K} \\ \mathfrak{m}_K^{n'} & \xrightarrow{\exp_\eta} & K^\times \end{array}$$

given by Theorem 4.11, we have that

$$\{\chi_L, U^{e \text{Sw } \chi - \delta_{\text{tor}}(L/K) + 1} \hat{K}_q(L)\}_k = \{\chi_L, U^{en' - \delta_{\text{tor}}(L/K) + en + 1} \hat{K}_q(L)\}_k = \{\chi, U_K^{\text{Sw } \chi + 1}\}_k = 0$$

but

$$\{\chi_L, U^{e \text{Sw } \chi - \delta_{\text{tor}}(L/K)} \hat{K}_q(L)\}_k = \{\chi_L, U^{en' - \delta_{\text{tor}}(L/K) + en} \hat{K}_q(L)\}_k = \{\chi, U_K^{\text{Sw } \chi}\}_k \neq 0.$$

This clearly yields  $\{\chi_L, U^{e \text{Sw } \chi - \delta_{\text{tor}}(L/K)} \hat{K}_q(L)\}_L \neq 0$ , so it remains to show that

$$\{\chi_L, U^{e \text{Sw } \chi - \delta_{\text{tor}}(L/K) + 1} \hat{K}_q(L)\}_L = 0.$$

Let  $r = \text{Sw } \chi$ , and  $\pi_K^{-r} \otimes \omega \in \mathfrak{m}_K^{-r} / \mathfrak{m}_K^{-r+1} \otimes_k \Omega_k^1(\log)$  denote Kato's refined Swan conductor. Elements of the form  $a\omega$ , where  $a \in k$ , generate  $\Omega_k^1(\log)$  over  $\mathbb{Z}$ , as well as elements of the form  $a\omega$ , where  $a \in L_0$ , generate  $\Omega_{L_0}^1(\log)$  over  $\mathbb{Z}$ . Then elements of the form  $a\omega \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}}$ , where  $a \in l$  and  $b_j \in l^\times$ , generate  $\Omega_l^1(\log)$  over  $\mathbb{Z}$ . Denote by  $\tilde{a} \in \mathcal{O}_L, \tilde{b}_j \in L^\times$  lifts of these elements to  $\mathcal{O}_L$ . If  $\{\chi_L, U^{e \text{Sw } \chi - \delta_{\text{tor}}(L/K) + 1} \hat{K}_q(L)\}_L \neq 0$ , then  $\{\chi_L, 1 + \tilde{a} \pi_L^{e \text{Sw } \chi - \delta_{\text{tor}}(L/K) + 1}, \tilde{b}_1, \dots, \tilde{b}_{q-1}\}_L$  coincides with the image of  $a\omega \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}}$  under  $\Omega_l^q(\log) \rightarrow H^q(L)\{p\}$ . Then, for  $a \in k$  and  $\tilde{a}$  a lift of  $a$  to  $\mathcal{O}_K$ ,  $\{\chi, 1 + \tilde{a} \pi_K^{r+1}\}_K$  coincides with the image of  $a\omega$  under  $\Omega_k^1(\log) \rightarrow H^1(K)\{p\}$ , so  $\{\chi, U_K^{r+1}\} \neq 0$ , a contradiction. It follows that  $\{\chi_L, U^{e \text{Sw } \chi - \delta_{\text{tor}}(L/K) + 1} \hat{K}_q(L)\}_L = 0$ . Therefore,

$$\text{Sw } \chi_L = e \text{Sw } \chi - \delta_{\text{tor}}(L/K). \quad \square$$

**Theorem 4.13.** *Let  $L/K$  be an extension of complete discrete valuation fields of mixed characteristic. Assume that  $K$  has perfect residue field of characteristic  $p > 0$ .*

Denote by  $e(L/K)$  the ramification index of  $L/K$ . Assume that  $\chi \in H^1(K)$  is such that

$$\text{Sw } \chi \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil.$$

Denote by  $\chi_L$  its image in  $H^1(L)$ . Then

$$\text{Sw } \chi_L = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L/K).$$

*Proof.* Following the same argument as [4, §10], we can assume that the residue field  $l$  of  $L$  is finitely generated over the residue field  $k$  of  $K$ . Since we have proven Proposition 4.12, it is enough to show that the present case can be reduced that of a  $q$ -dimensional local field that is a finite extension of  $K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ .

Since  $l$  is finitely generated over  $k$ , there are  $T_1, \dots, T_{q-1} \in l$  such that  $l$  is a finite, separable extension of  $k(T_1, \dots, T_{q-1})$ . Since there is an embedding  $k(T_1, \dots, T_{q-1}) \hookrightarrow k((T_1)) \cdots ((T_{q-1}))$ , there is also an embedding  $l \hookrightarrow E$  of  $l$  into a finite, separable extension  $E$  of  $k((T_1)) \cdots ((T_{q-1}))$ . Since  $\{T_1, \dots, T_{q-1}\}$  is a  $p$ -basis for both  $l$  and  $E$ , there is a complete, discrete valuation field  $L(E)$  that is an extension of  $L$  satisfying  $\mathcal{O}_L \subset \mathcal{O}_{L(E)}$ ,  $\mathfrak{m}_L \subset \mathfrak{m}_{L(E)}$ ,  $\pi_L$  is still prime in  $L(E)$ , and the residue field of  $L(E)$  is isomorphic to  $E$  over  $l$ .

$L(E)$  is a finite extension of  $K\{\{T_1\}\} \cdots \{\{T_{q-1}\}\}$ . Further, since  $\delta_{\text{tor}}(L(E)/L) = 0$  and  $e(L(E)/L) = 1$ , we have  $e(L(E)/K) = e(L/K)$ . From Lemma 4.3,  $\delta_{\text{tor}}(L(E)/K) = \delta_{\text{tor}}(L/K)$ . From [4, Lemma 6.2], since  $\mathcal{O}_L \subset \mathcal{O}_{L(E)}$ ,  $\mathfrak{m}_{L(E)} = \mathcal{O}_{L(E)}\mathfrak{m}_L$ , and the extension of residue fields is separable, we have  $\text{Sw } \chi_{L(E)} = \text{Sw } \chi_L$ . Thus it is sufficient to prove that

$$\text{Sw } \chi_{L(E)} = e(L/K) \text{Sw } \chi - \delta_{\text{tor}}(L(E)/K),$$

which follows from Proposition 4.12. □

## 5 A generalized $\psi$ -function

Through this section, let  $L/K$  be an extension of complete discrete valuation fields such that the residue field of  $K$  is perfect and of characteristic  $p > 0$ . We define generalizations of the classical  $\psi$ -function for this case. More precisely, we will define functions  $\psi_{L/K}^{\text{AS}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  and  $\psi_{L/K}^{\text{ab}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  and show that, in the classical case of  $L/K$  finite, they both coincide with the classical  $\psi_{L/K} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  (see Theorem 5.5). The superscripts AS and ab refer, respectively, to Abbes-Saito and abelian. In the definition of  $\psi_{L/K}^{\text{AS}}$  we use the Abbes-Saito upper ramification filtrations of absolute Galois groups, while in the definition of  $\psi_{L/K}^{\text{ab}}$  we use Kato's ramification filtration of  $H^1(L)$ .

We also define functions  $\varphi_{L/K}^{\text{AS}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  and  $\varphi_{L/K}^{\text{ab}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  and show that, when  $\varphi_{L/K}^{\text{AS}}$  and  $\varphi_{L/K}^{\text{ab}}$  are injective,  $\psi_{L/K}^{\text{AS}}$  and  $\psi_{L/K}^{\text{ab}}$  are their respective left inverses (and vice-versa).

Assume first that the residue field  $k$  of  $K$  is algebraically closed. For  $t \in \mathbb{Z}_{(p)}$ ,  $t \geq 0$ , define



$\psi_{L/K}^{\text{ab}}(t) \in \mathbb{R}_{\geq 0}$  as

$$\psi_{L/K}^{\text{ab}}(t) = \inf \left\{ s \in \mathbb{Z}_{(p)} \left| \begin{array}{l} \text{Im}(F_{e(K'/K)t} H^1(K') \rightarrow H^1(LK')) \subset F_{e(LK'/L)s} H^1(LK') \text{ for} \\ \text{all finite extensions } K'/K \text{ of complete discrete valuation} \\ \text{fields such that } K'/K \text{ is tame and } e(LK'/L)s, e(K'/K)t \in \mathbb{Z} \end{array} \right. \right\},$$

and then extend  $\psi_{L/K}^{\text{ab}}$  to  $\mathbb{R}_{\geq 0}$  by putting

$$\psi_{L/K}^{\text{ab}}(t) = \sup \{ \psi_{L/K}^{\text{ab}}(s) : s \leq t, s \in \mathbb{Z}_{(p)} \}.$$

Similarly, for  $t \in \mathbb{Z}_{(p)}$ ,  $t \geq 0$ , define  $\varphi_{L/K}^{\text{ab}}(t) \in \mathbb{R}_{\geq 0}$  as

$$\varphi_{L/K}^{\text{ab}}(t) = \sup \left\{ s \in \mathbb{Z}_{(p)} \left| \begin{array}{l} \text{Im}(F_{e(K'/K)s} H^1(K') \rightarrow H^1(LK')) \subset F_{e(LK'/L)t} H^1(LK') \text{ for} \\ \text{all finite extensions } K'/K \text{ of complete discrete valuation} \\ \text{fields such that } K'/K \text{ is tame and } e(LK'/L)t, e(K'/K)s \in \mathbb{Z} \end{array} \right. \right\},$$

and then extend  $\varphi_{L/K}^{\text{ab}}$  to  $\mathbb{R}_{\geq 0}$  by putting

$$\varphi_{L/K}^{\text{ab}}(t) = \sup \{ \varphi_{L/K}^{\text{ab}}(s) : s \leq t, s \in \mathbb{Z}_{(p)} \}.$$

Let  $G_{K, \log}^{t+}$  denote the Abbes-Saito logarithmic upper ramification filtration defined in [1]. We now define  $\psi_{L/K}^{\text{AS}}$  and  $\varphi_{L/K}^{\text{AS}}$  by putting, for  $t \in \mathbb{R}_{\geq 0}$ ,

$$\psi_{L/K}^{\text{AS}}(t) = \inf \{ s \in \mathbb{R} : \text{Im}(G_{L, \log}^{s+} \rightarrow G_K) \subset G_{K, \log}^{t+} \}$$

and

$$\varphi_{L/K}^{\text{AS}}(t) = \sup \{ s \in \mathbb{R} : \text{Im}(G_{L, \log}^{t+} \rightarrow G_K) \subset G_{K, \log}^{s+} \}.$$

When  $k$  is not necessarily algebraically closed, we define  $\psi_{L/K}^{\text{ab}}$ ,  $\varphi_{L/K}^{\text{ab}}$ ,  $\psi_{L/K}^{\text{AS}}$  and  $\varphi_{L/K}^{\text{ab}}$  as follows. Let  $\tilde{K} = \widehat{K_{\text{ur}}}$  and  $\tilde{L} = \widehat{LK_{\text{ur}}}$ . Then define  $\psi_{L/K}^{\text{ab}} = \psi_{\tilde{L}/\tilde{K}}^{\text{ab}}$ ,  $\varphi_{L/K}^{\text{ab}} = \varphi_{\tilde{L}/\tilde{K}}^{\text{ab}}$ ,  $\psi_{L/K}^{\text{AS}} = \psi_{\tilde{L}/\tilde{K}}^{\text{AS}}$  and  $\varphi_{L/K}^{\text{AS}} = \varphi_{\tilde{L}/\tilde{K}}^{\text{AS}}$ .

The above defined functions have properties similar to those of their classical counterparts. We will now prove some of these properties.

**Proposition 5.1.** *If  $\varphi_{L/K}^{\text{ab}}(t)$  is injective, then  $\psi_{L/K}^{\text{ab}}(t)$  is its left inverse. Similarly, if  $\psi_{L/K}^{\text{ab}}(t)$  is injective, then  $\varphi_{L/K}^{\text{ab}}(t)$  is its left inverse.*

*Proof.* From the definitions of  $\varphi_{L/K}^{\text{ab}}(t)$  and  $\psi_{L/K}^{\text{ab}}(t)$ , we can assume that  $k$  is algebraically closed. We shall prove that if  $\varphi_{L/K}^{\text{ab}}(t)$  is injective, then  $\psi_{L/K}^{\text{ab}}(t)$  is its left inverse. The other statement is proved in an analogous way.

It's enough to show that, for  $t \in \mathbb{Z}_{(p)}$ ,  $t \geq 0$ , we have  $t = \psi_{L/K}^{\text{ab}}(\varphi_{L/K}^{\text{ab}}(t))$ . If  $s \in \mathbb{Z}_{(p)}$  is smaller or equal to  $\varphi_{L/K}^{\text{ab}}(t)$ , then

$$\text{Im}(F_{e(K'/K)s}H^1(K') \rightarrow H^1(LK')) \subset F_{e(LK'/L)t}H^1(LK')$$

for all finite Galois extensions  $K'/K$  of complete discrete valuation fields such that  $K'/K$  is tame and  $e(LK'/L)t, e(K'/K)s \in \mathbb{Z}$ . Then  $t \geq \psi_{L/K}^{\text{ab}}(s)$ , so  $t \geq \psi_{L/K}^{\text{ab}}(\varphi_{L/K}^{\text{ab}}(t))$ .

Assume that we have  $t > \psi_{L/K}^{\text{ab}}(\varphi_{L/K}^{\text{ab}}(t))$ . Take  $\tilde{t} \in \mathbb{Z}_{(p)}$  such that  $\psi_{L/K}^{\text{ab}}(\varphi_{L/K}^{\text{ab}}(t)) < \tilde{t} < t$ . Let  $K'/K$  be any finite Galois extension of complete discrete valuation fields that is tame and such that  $e(LK'/L)\tilde{t} \in \mathbb{Z}$ . Since  $\tilde{t} > \psi_{L/K}^{\text{ab}}(\varphi_{L/K}^{\text{ab}}(t))$ ,

$$\text{Im}(F_{e(K'/K)s}H^1(K') \rightarrow H^1(LK')) \subset F_{e(LK'/L)\tilde{t}}H^1(LK')$$

for every  $s \leq \varphi_{L/K}^{\text{ab}}(t)$  in  $\mathbb{Z}_{(p)}$  such that  $e(K'/K)s \in \mathbb{Z}$ . Then

$$\varphi_{L/K}^{\text{ab}}(\tilde{t}) \geq \varphi_{L/K}^{\text{ab}}(t).$$

Since  $\varphi_{L/K}^{\text{ab}}$  is clearly increasing and  $t > \tilde{t}$ , we get

$$\varphi_{L/K}^{\text{ab}}(\tilde{t}) = \varphi_{L/K}^{\text{ab}}(t),$$

which contradicts the injectivity assumption. Therefore

$$t = \psi_{L/K}^{\text{ab}}(\varphi_{L/K}^{\text{ab}}(t))$$

for every  $t \geq 0$  and we conclude that  $\psi_{L/K}^{\text{ab}}(t)$  is the left inverse of  $\varphi_{L/K}^{\text{ab}}(t)$ .  $\square$

The analogous result for  $\psi_{L/K}^{\text{AS}}$  and  $\varphi_{L/K}^{\text{AS}}$  is also true:

**Proposition 5.2.** *If  $\varphi_{L/K}^{\text{AS}}(t)$  is injective, then  $\psi_{L/K}^{\text{AS}}(t)$  is its left inverse. Similarly, if  $\psi_{L/K}^{\text{AS}}(t)$  is injective, then  $\varphi_{L/K}^{\text{AS}}(t)$  is its left inverse.*

*Proof.* From the definitions of  $\varphi_{L/K}^{\text{AS}}(t)$  and  $\psi_{L/K}^{\text{AS}}(t)$ , we can assume that  $k$  is algebraically closed. We shall prove that if  $\varphi_{L/K}^{\text{AS}}(t)$  is injective, then  $\psi_{L/K}^{\text{AS}}(t)$  is its left inverse. The other statement is proved in an analogous way.

If  $s \in \mathbb{R}$  is less than or equal to  $\varphi_{L/K}^{\text{AS}}(t)$ , then

$$\text{Im}(G_{L, \log}^{t+} \rightarrow G_K) \subset G_{K, \log}^{s+}.$$

Hence  $t \geq \psi_{L/K}^{\text{AS}}(s) \geq \psi_{L/K}^{\text{AS}}(\varphi_{L/K}^{\text{AS}}(t))$ .

Assume that we have  $t > \psi_{L/K}^{\text{AS}}(\varphi_{L/K}^{\text{AS}}(t))$ . Take  $\tilde{t} \in \mathbb{R}$  such that  $\psi_{L/K}^{\text{AS}}(\varphi_{L/K}^{\text{AS}}(t)) < \tilde{t} < t$ . Then

$$\text{Im}(G_{L, \log}^{\tilde{t}+} \rightarrow G_K) \subset G_{K, \log}^{s+}$$

for every  $s \leq \varphi_{L/K}^{\text{AS}}(t)$ . Thus

$$\varphi_{L/K}^{\text{AS}}(\tilde{t}) \geq \varphi_{L/K}^{\text{AS}}(t).$$

Since  $\varphi_{L/K}^{\text{AS}}$  is clearly increasing and  $t > \tilde{t}$ , we get

$$\varphi_{L/K}^{\text{AS}}(\tilde{t}) = \varphi_{L/K}^{\text{AS}}(t),$$

which contradicts the injectivity assumption. Therefore

$$t = \psi_{L/K}^{\text{AS}}(\varphi_{L/K}^{\text{AS}}(t))$$

for every  $t \geq 0$  and we conclude that  $\psi_{L/K}^{\text{AS}}(t)$  is the left inverse of  $\varphi_{L/K}^{\text{AS}}(t)$ .  $\square$

These functions satisfy formulas similar to those satisfied by the classical  $\varphi$  and  $\psi$ -functions, as we can see from the following lemma.

**Lemma 5.3.** *Let  $K'$  be a finite Galois extension of  $K$  that is tamely ramified and  $L' = LK'$ . Then*

$$\begin{aligned}\varphi_{L'/K'}^{\text{ab}}(e(L'/L)t) &= e(K'/K)\varphi_{L/K}^{\text{ab}}(t), \\ \psi_{L'/K'}^{\text{ab}}(e(K'/K)t) &= e(L'/L)\psi_{L/K}^{\text{ab}}(t), \\ \varphi_{L'/K'}^{\text{AS}}(e(L'/L)t) &= e(K'/K)\varphi_{L/K}^{\text{AS}}(t), \\ \psi_{L'/K'}^{\text{AS}}(e(K'/K)t) &= e(L'/L)\psi_{L/K}^{\text{AS}}(t).\end{aligned}$$

*Proof.* Follows from the definitions. For example,

$$\begin{aligned}\varphi_{L'/K'}^{\text{AS}}(e(L'/L)t) &= \sup \left\{ s \in \mathbb{R} : \text{Im}(G_{L,\log}^{e(L'/L)t+} \rightarrow G'_K) \subset G_{K',\log}^{s+} \right\} \\ &= \sup \left\{ s \in \mathbb{R} : \text{Im}(G_{L,\log}^{t+} \rightarrow G_K) \subset G_{K,\log}^{\overline{\frac{s}{e(K'/K)}}+} \right\} \\ &= e(K'/K) \sup \left\{ s \in \mathbb{R} : \text{Im}(G_{L,\log}^{t+} \rightarrow G_K) \subset G_{K,\log}^{s+} \right\} \\ &= e(K'/K)\varphi_{L/K}^{\text{AS}}(t).\end{aligned}$$

$\square$

We relate this section with the rest of our paper. The main results that we proved in the previous sections are, in reality, results about  $\psi_{L/K}^{\text{ab}}$ . More precisely, we have the following theorem:

**Theorem 5.4.** *Let  $L/K$  be an extension of complete discrete valuation fields. Assume that  $K$  has perfect residue field of characteristic  $p > 0$ . Let  $t \in \mathbb{R}_{\geq 0}$  be such that*

$$\begin{cases} t \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil & \text{if } K \text{ is of characteristic } 0, \\ t > \frac{p}{p-1} \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} & \text{if } K \text{ is of characteristic } p. \end{cases}$$

Then

$$\psi_{L/K}^{\text{ab}}(t) = e(L/K)t - \delta_{\text{tor}}(L/K).$$

*Proof.* Write

$$T(L/K) = \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil$$

when  $K$  is of characteristic 0, and

$$T(L/K) = \frac{p}{p-1} \frac{\delta_{\text{tor}}(L/K)}{e(L/K)}$$

when  $K$  is of characteristic  $p$ . Let  $t \in \mathbb{R}_{\geq 0}$  be such that  $t \geq T(L/K)$  if  $K$  is of characteristic 0 and  $t > T(L/K)$  if  $K$  is of characteristic  $p$ .

If  $t \in \mathbb{Z}$ , it follows from Theorems 2.11 and 4.13 that

$$\psi_{L/K}^{\text{ab}}(t) = e(L/K)t - \delta_{\text{tor}}(L/K).$$

If  $t \in \mathbb{Z}_{(p)}$ , take a finite Galois extension  $K'/K$  that is tamely ramified and such that  $e(K'/K)t \in \mathbb{Z}$ . Observe that, if  $K$  is of characteristic 0,

$$\begin{aligned} e(K'/K)T(L/K) &= \frac{2e_{K'}}{p-1} + \frac{e(L'/L)}{e(L'/K')} + e(K'/K) \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil \\ &\geq \frac{2e_{K'}}{p-1} + \frac{e(L'/L)}{e(L'/K')} + \left\lceil \frac{e(L'/L)\delta_{\text{tor}}(L/K)}{e(L'/K')} \right\rceil \\ &\geq \frac{2e_{K'}}{p-1} + \frac{1}{e(L'/K')} + \left\lceil \frac{\delta_{\text{tor}}(L'/K')}{e(L'/K')} \right\rceil = T(L'/K'). \end{aligned}$$

Similarly, if  $K$  is of characteristic  $p$ ,

$$\begin{aligned} e(K'/K)T(L/K) &= \frac{p}{p-1} \frac{e(K'/K)\delta_{\text{tor}}(L/K)}{e(L/K)} \\ &= \frac{p}{p-1} \frac{e(L'/L)\delta_{\text{tor}}(L/K)}{e(L'/K')} \\ &= \frac{p}{p-1} \frac{\delta_{\text{tor}}(L'/K')}{e(L'/K')} = T(L'/K'). \end{aligned}$$

Then  $e(K'/K)t \geq T(L'/K')$  if  $K$  is of characteristic 0 and  $e(K'/K)t > T(L'/K')$  if  $K$  is of characteristic  $p$ . It follows that

$$\begin{aligned} \psi_{L'/K'}^{\text{ab}}(e(K'/K)t) &= e(L'/K')e(K'/K)t - \delta_{\text{tor}}(L'/K') \\ &= e(L'/L)e(L/K)t - e(L'/L)\delta_{\text{tor}}(L/K). \end{aligned}$$

From Lemma 5.3, we conclude that

$$\psi_{L/K}^{\text{ab}}(t) = \frac{\psi_{L'/K'}^{\text{ab}}(e(K'/K)t)}{e(L'/L)} = e(L/K)t - \delta_{\text{tor}}(L/K).$$

The result then follows from the definition of  $\psi_{L/K}^{\text{ab}}$ . □

In the classical case, the functions we defined in fact coincide with the classical  $\varphi$  and  $\psi$ -functions, as is shown in the following theorem.

**Theorem 5.5.** *If  $L/K$  is a finite Galois extension and  $k$  is perfect, we have*

$$\psi_{L/K} = \psi_{L/K}^{\text{ab}} = \psi_{L/K}^{\text{AS}}.$$

*Proof.* From the definitions of the functions, we can assume that  $k$  is algebraically closed. We shall first show that  $\psi_{L/K}^{\text{AS}} = \psi_{L/K}$ . To show that  $\varphi_{L/K}(t) \leq \varphi_{L/K}^{\text{AS}}(t)$ , just observe that, if  $L'/L$  is a finite Galois extension over  $K$ , then

$$G(L'/L)^t = G(L'/L)_{\psi_{L'/K} \circ \varphi_{L/K}(t)} \subset G(L'/K)_{\psi_{L'/K} \circ \varphi_{L/K}(t)} = G(L'/K)^{\varphi_{L/K}(t)}.$$

Since the Abbes-Saito filtration is left continuous with rational jumps, it remains to show that  $\varphi_{L/K}(t) \geq \varphi_{L/K}^{\text{AS}}(t)$  for  $t \in \mathbb{Q}_{\geq 0}$ . Let  $K'$  be a finite Galois extension of  $K$  that is tame and write  $L' = LK'$ . Since  $L'/L$  and  $K'/K$  are tame extensions, we have

$$\begin{aligned} \varphi_{L'/K'}(e(L'/L)t) &= e(K'/K)\varphi_{K'/K} \circ \varphi_{L'/K'} \circ \psi_{L'/L}(t) \\ &= e(K'/K)\varphi_{L'/K} \circ \psi_{L'/L}(t) \\ &= e(K'/K)\varphi_{L/K}(t). \end{aligned}$$

From Serre's local class field theory for fields with algebraically closed residue field ([8]), for every  $s \in \mathbb{Z}_{\geq 0}$ , the maps

$$\frac{(G_{L'}^{\text{ab}})^{\psi_{L'/K'}(s)}}{(G_{L'}^{\text{ab}})^{\psi_{L'/K'}(s)+1}} \rightarrow \frac{(G_{K'}^{\text{ab}})^s}{(G_{K'}^{\text{ab}})^{s+1}}$$

have images that are of finite index and nontrivial. Taking  $K'$  such that  $e(K'/K)\varphi_{L/K}(t)$  is an integer and setting  $s = \varphi_{L'/K'}(e(L'/L)t)$ , we see that the image of  $(G_{L'}^{\text{ab}})^{e(L'/L)t} = (G_L^{\text{ab}})^t$  is not contained in  $(G_{K'}^{\text{ab}})^{\varphi_{L'/K'}(e(L'/L)t)+1}$ .

Since  $(G_{K'}^{\text{ab}})^{e(K'/K)\varphi_{L/K}(t)+1} = (G_K^{\text{ab}})^{\varphi_{L/K}(t) + \frac{1}{e(K'/K)}}$ , we have that the image of  $(G_L^{\text{ab}})^t$  is not contained in  $(G_K^{\text{ab}})^{\varphi_{L/K}(t) + \frac{1}{e(K'/K)}}$ . We can choose tame extensions with  $e(K'/K)$  arbitrarily large, so we have that  $\varphi_{L/K}(t) \geq \varphi_{L/K}^{\text{AS}}(t)$ . Hence  $\psi_{L/K}^{\text{AS}} = \psi_{L/K}$ .

Now we shall prove that  $\psi_{L/K}^{\text{ab}} = \psi_{L/K}$ . Let  $K'/K$  be a finite, separable extension of complete discrete valuation fields that is tamely ramified and such that  $e(LK'/L)t$  and  $e(K'/K)\varphi_{L/K}(t)$  are integers. Write  $L' = LK'$ . Observe that, taking into account that  $K'/K$  and  $L'/L$  are tamely ramified, we have

$$\begin{aligned} \psi_{L'/K'}(e(K'/K)\varphi_{L/K}(t)) &= \psi_{L'/K'} \circ \psi_{K'/K} \circ \varphi_{L/K}(t) = \psi_{L'/K} \circ \varphi_{L/K}(t) \\ &= \psi_{L'/L} \circ \psi_{L/K} \circ \varphi_{L/K}(t) = \psi_{L'/L}(t) = e(L'/L)t. \end{aligned}$$

Let

$$\chi \in F_{e(K'/K)\varphi_{L/K}(t)} H^1(K').$$

Denote by  $\chi_{L'}$  its image in  $H^1(L')$ . Using the same argument as before we see that

$$\chi_{L'} \in F_{e(L'/L)t} H^1(L'),$$

so  $\varphi_{L/K}(t) \leq \varphi_{L/K}^{\text{ab}}(t)$ . Now, if  $s = \varphi_{L/K}(t) + \frac{1}{e(K'/K)}$ , then

$$F_{e(K'/K)s}H^1(K') = F_{e(K'/K)\varphi_{L/K}(t)+1}H^1(K').$$

Since the image of  $F_{e(K'/K)\varphi_{L/K}(t)+1}H^1(K')$  is not contained in  $F_{e(L'/L)t}H^1(L')$ , we have that  $s > \varphi_{L/K}^{\text{ab}}(t)$ . Since we can take extensions  $K'/K$  with arbitrarily large  $e(K'/K)$ , we get that  $\varphi_{L/K} = \varphi_{L/K}^{\text{ab}}$ . Thus  $\psi_{L/K}^{\text{ab}} = \psi_{L/K}$ .  $\square$

The properties we proved and Theorem 5.5 give evidence that the above defined functions  $\psi_{L/K}^{\text{ab}}$  and  $\psi_{L/K}^{\text{AS}}$  are good generalizations of the classical  $\psi$ -function. We can conjecture:

**Conjecture 1.** *Let  $L/K$  be an extension of complete discrete valuation fields. Assume that  $k$  is algebraically closed of characteristic  $p > 0$ . Then*

$$\psi_{L/K}^{\text{ab}} = \psi_{L/K}^{\text{AS}}.$$

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